Name:____________________ ________________________

1	2	3	4	5	6	7	*Total*

Math 256, Section 11, Spring 2003

Final Exam

All problems are worth 10 points, but some are harder than the others.

Justify your answers carefully. You are allowed to use homework problems and facts from the book or from the lectures in your proofs.

Good luck!

1. Mark the following statements 'T' for True or 'F' for False. No justification is necessary.

(a) The extension $\mathbb{Q}(\pi^2) \subset \mathbb{Q}(\pi)$ is finite.

(b) The extension $\mathbb{Q}(\pi^2) \subset \mathbb{Q}(\pi)$ is normal.

(c) For any field F and any irreducible polynomial $P \in F[x]$, all roots of P in $\overline{F}$ are simple.

(d) The Galois group of a simple normal extension of fields is a simple group.

(e) If A is a square matrix with real entries and $z = a + bi \in \mathbb{C}$ is an eigenvalue of A, then $\overline{z} = a - ib \in \mathbb{C}$ also is an eigenvalue of A.

2. The equation $x^3 + x + 1 = 0$, $x \in \mathbb{R}$ has exactly one solution $\alpha \in \mathbb{R}$, which is irrational (you do not need to prove it). Consider $E = \mathbb{Q}(\alpha)$.

(a) Find $[E : \mathbb{Q}]$. (Some justification is required for full credit).

(b) Write a basis for E over $\mathbb{Q}$ in terms of α.

3. Give an example of a field extension $F \supset \mathbb{Z}_p$ that is algebraic, but not finite.

4. A 3×3 matrix A has only two eigenvalues in $\mathbb{R}$: 0 and 1.

(a) What is the characteristic polynomial of A (two answers)?

(b) Compute $A^2(A - I)^2$.

(c) Write the Jordan form of A (there are four significantly different answers).

5. Let $E = \mathbb{Q}(\alpha)$ be the extension from Problem 2: so $\alpha \in \mathbb{R}$ is the unique real root of $x^3 + x + 1$ (you can use the fact that α is irrational).

(a) Is the extension $E \supset \mathbb{Q}$ normal?

(c) Let K be the splitting field of $x^3 + x + 1$ over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

6. Prove that the splitting fields of

$$(x^{p^2} - x)(x^{p^3} - x) \in \mathbb{Z}_p[x]$$

and

$$x^{p^6} - x \in \mathbb{Z}_p[x]$$

over $\mathbb{Z}_p$ coincide. Here p is a prime number.

7. Let $\mathbb{Q}(\alpha) \supset \mathbb{Q}$ be a normal extension whose Galois group $G(\mathbb{Q}(\alpha)/\mathbb{Q}) = \{id, \phi, \phi^2, \phi^3\}$ is the cyclic group of order 4 generated by $\phi : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$.

(a) Prove that the minimal polynomial of α over $\mathbb{Q}$ is $(x - \alpha)(x - \phi(\alpha))(x - \phi^2(\alpha))(x - \phi^3(\alpha))$.

(b) Show that there is exactly one subfield $E \subset \mathbb{Q}(\alpha)$ such that $E \neq \mathbb{Q}(\alpha)$, $E \neq \mathbb{Q}$.

(c) Show that $\beta = \alpha + \phi^2(\alpha)$ and $\gamma = \alpha \times \phi^2(\alpha)$ belong to E (in fact, $E = \mathbb{Q}(\beta, \gamma)$, but you do not need to prove it).

Additional practice problems on Part X.

8. Find the smallest normal extension of $\mathbb{Q}$ that contains $\sqrt{1+\sqrt{2}}$.

9. How many elements are there in the Galois group of x^4-5x^2+6 over $\mathbb{Q}$ (the Galois group of $P \in F[x]$ over F is the Galois group of its splitting field).

10. Let $E \supset F$ be a finite normal extension. Is it true that E is the splitting field of some polynomial $P \in F[x]$ over F?

11. Give an example of three fields $K \supset E \supset F$ such that K is a splitting field over E, E is a splitting field over F, but K is not a splitting field over F.

12. Give an example of an irreducible polynomial $P \in \mathbb{Q}[x]$ whose Galois group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (hint: start with the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \supset \mathbb{Q}$).

13. Is the Galois group of x^4-2x^2-1 over $\mathbb{Q}$ abelian? How many elements does it have?

14. Let E be the splitting field of a polynomial $P \in F[x]$ over F. Suppose that $[E : F] = (\deg(F))!$; show that P is irreducible.

15. F is a field with algebraic closure $\overline{F}$. You are told that $F(\alpha)$ and $F(\beta)$ are splitting fields over F, where $\alpha, \beta \in \overline{F}$. Prove that $F(\alpha, \beta)$ is also a splitting field over F.

16. Let $E \supset F$ be a finite extension with the property that $E_{G(E/F)} = F$ (that is, the only elements of E that satisfy $\phi(e) = e$ for all automorphisms of E over F are elements of F). Show that E is a splitting field over F.

(Hint: for an element $a \in E$, use the polynomial

$$\prod_{\sigma \in G(E/F)} (x - \sigma(a))$$

to prove that all conjugates of a over F are in E. With a little more work, this approach also shows $E \supset F$ is separable.)