1. Set

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A and the corresponding eigenspaces.

The characteristic polynomial is

$$\det \begin{pmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 & -1 \\ 0 & 1 & \lambda + 1 \end{pmatrix} = (\lambda - 1)((\lambda - 1)(\lambda + 1) - 1) = \lambda^2(\lambda - 1),$$

so the eigenvalues are $\lambda = 0$ and $\lambda = 1$.

The eigenspace corresponding to $\lambda = 0$ is the kernel $\{(x, y, z) : (x, y, z)A = (0, 0, 0)\}$. Solving the system of linear equations, we find that it equals $\{(0, y, y)\} = Span\langle(0, 1, 1)\rangle$. Similarly, the eigenspace corresponding to $\lambda = 1$ equals $\{(x, y, z) : (x, y, z)A = (x, y, z)\} = \{(x, x, x)\} = Span\langle(1, 1, 1)\rangle$.

(b) Find the Jordan form of A. Justify.

Since the characteristic polynomial is a product of linear factors, the Jordan form exists. The only eigenvalues are 0 and 1, so the Jordan blocks must have either 0 or 1 on the diagonal. The eigenvalue 0 has algebraic multiplicity 2 and geometric multiplicity 1, so there is a single Jordan block of size 2×2 for this eigenvalue. The eigenvalue 1 has algebraic and geometric multiplicity 1, which gives us a single 1×1 Jordan block. Therefore, the Jordan form of A is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. A is a 3×3 matrix over $\mathbb{Q}$ that has exactly two eigenvalues: 2 and 3.

(a) What is the characteristic polynomial of A (find all answers for full credit)?

The characteristic polynomial is a monic cubic polynomial with only two roots: 2 and 3. This means it is either $(x-2)^2(x-3)$ or $(x-2)(x-3)^2$.

(b) Find a non-zero polynomial $p(x)$ of degree 4 such that $p(A) = 0$. Your polynomial has to work for all such matrices. Explain.

By the Cayley-Hamilton Theorem, any polynomial that is divisible by the characteristic polynomial would work. We can therefore take $p(x) = (x-2)^2(x-3)^2$: we see from part (a) that it is divisible by the characteristic polynomial.

3. A is an $n \times n$ matrix over $\mathbb{C}$. Prove that the following two conditions are equivalent:

(1) For every eigenvalue λ of A, we have

$$\dim\{v \in \mathbb{C}^n : vA = \lambda v\} = 1;$$

(2) The characteristic polynomial of A is the minimal polynomial of A.

The characteristic polynomial of A is a product of linear factors, because A is over $\mathbb{C}$:

$$\det(\lambda I - A) = (\lambda - \lambda_1)^n_1 \cdot \ldots (\lambda - \lambda_m)^{n_m}.$$

Therefore, the Jordan form of A exists. In terms of the Jordan form, the condition (1) says that there is a single Jordan block for each eigenvalue, while the condition (2) says that for each eigenvalue λ_i, the size of the largest Jordan block is n_i. The two conditions are equivalent, because the size of all Jordan blocks with eigenvalue λ_i adds up to n_i.

4. Let $V = \{a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 : a_i \in \mathbb{R}\}$ be the space of polynomials (with real coefficients) of degree 4 or less. Define a map $\phi : V \to V$ by

$$(\phi(p))(x) = p(x+1);$$

for instance, $\phi(x^2 - 3x + 1) = (x+1)^2 - 3(x+1) + 1 = x^2 - x - 1$.

(a) Find the minimal polynomial of ϕ.

This could be done by writing the matrix of ϕ. Another way is to guess the polynomial: $(x-1)^5$ and then prove that its minimal. Let us do it.

Consider the map $\phi - I$. It is easy to see that for any $p(x) \in V$, $\deg((\phi - I)p) = \deg(p) - 1$, unless $p(x)$ is a constant polynomial, and then $(\phi - I)p = 0$. This follows from the observation that in $(\phi - I)p = p(x+1) - p(x)$, the leading term cancels out, but the next term does not. Therefore, $(\phi - I)^5 = 0$ (after 5 iterations, any polynomial of degree at most 4 vanishes), but $(\phi - I)^4 \neq 0$ (if we start with a polynomial of degree 4, then after 4 iterations we obtain a non-zero constant). This means that the minimal polynomial divides $(x-1)^5$, but not $(x-1)^4$; so we see that the minimal polynomial is $(x-1)^5$.

(b) Does the Spectral Theorem apply to ϕ? If no, explain why; if yes, explain what it implies for ϕ.

Since the minimal polynomial is a product of linear factors, the Spectral Theorem applies to ϕ. It claims that V is the interior direct sum of a single space $W = \ker(\phi - I)^5$, that is, $V = W$. However, this is obvious because the polynomial $(x-1)^5$ annihilates ϕ. To summarize, the theorem works but it is trivial.

5. A vector space V (over some field F) has dimension 5. Suppose $\phi : V \to V$ is a linear map, and $W_1, W_2 \subset V$ are stable subspaces: $\phi(W_1) \subseteq W_1$, $\phi(W_2) \subseteq W_2$. The Jordan forms of the restrictions $\phi|_{W_1}$ and $\phi|_{W_2}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad .$$

Find the Jordan form of ϕ. Justify your answer.

Answer:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Let us find the minimal polynopmial $p(x)$ of ϕ. Note that $\deg p(x) \leq 5$, because $\dim(V) = 5$. Besides, $p(\phi) = 0$ implies that $p(\phi|_{W_i}) = 0$ for $i = 1, 2$, so that $p(x)$ is divisible by the minimal polynomials of $\phi|_{W_1}$, $\phi|_{W_2}$. The minimal polynomials are easily seen from the Jordan forms: $(x-1)(x-2)(x-3)$ and $(x-1)(x-4)^2$. This implies that $p(x) = (x-1)(x-2)(x-3)(x-4)^2$; therefore, the jordan form of ϕ is as above.