$S^+, H^+ \subset C(X)$ cones, $S^+_x, H^+_x$ the bases for the cones.

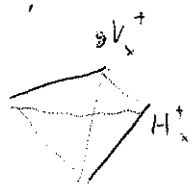
RRT: $s \in S^+$, $\exists!\ \overset{V^+}{\overset{\in}{k}}, \overset{H^+}{\overset{\in}{h}}$ s.t. $s = gk + h$.

furthermore: $k = (1-P)s$, $h = \lim P^n s$.

Cor: - an extreme point of $S^+_x$ (extreme ray of $S^+$) is either of the form $gk$ or $h$.

- in the first case $k$ must be of the form $\delta_y$ thus $gk$ normalized to $S^+_x$ is $g(y,\cdot)/g(y,x)$

- in the second case $h$ is an extreme point of $H^+$, that is, a minimal harmonic function.




( Def: $h \in H^+$ is minimal if $h' \in H^+$, $h' \leq h \Rightarrow h' = \lambda h$.
observation: $h \in H^+_x$ is minimal $\Leftrightarrow$ it is extreme )

$$\begin{tikzcd} X \arrow[r, "i_x"] \arrow[rr, bend left, "j_x"] & C(X) \arrow[r, "e^{-y}"] & C(X, \mathbb{R}^*_+) \subset C(X) \end{tikzcd}$$

$$y \longmapsto \rho(y,\cdot) - \rho(y,x) \longmapsto \frac{f(y,\cdot)}{f(y,x)} = \frac{g(y,\cdot)}{g(y,x)}$$

thus the potential extreme points are $j_x(X)$.

KM $\Rightarrow$ there are $H^+_x$ extreme points as well.

next we'll show: $\quad j_x$

$$\overline{\mathrm{conv}(j_x(X))} = S^+_x$$

$$\Rightarrow \quad \mathrm{extreme}(S^+_x) \subset \overline{j_x(X)} = \mathrm{Martin}(X).$$

- $D \subset X$ finite

- $L_D(y) = P_y\{(z) \mid \overset{i\geq 0}{z_i \notin D}\}$, for $y \in D$.

- $\tau : \Omega = X^\infty \longrightarrow \mathbb{Z}_+$, $\tau((z)) = \sup\{n \mid z_n \in D\}$ time of leaving $D$.

- $P_x\{(z) \mid z_{\tau(z)} = y\} = \sum_m P_x\{\tau = m, z_m = y\} = \sum_m p^m(x,y)\cdot L_D(y) = g(x,y)\cdot L_D(y)$

$$\Rightarrow P_x\{(z) \mid z_\tau = y\} = g(x,y) L_D(y)$$

- $P_x\{X_\tau = y\} = g(x,y)L_D(y) = \frac{g(x,y)}{g(x_0,y)}\cdot g(x_0,y)L_D(y) = \frac{g(x,y)}{g(x_0,y)} P_{x_0}(X_\tau = y)$

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By the Super-Martingale convergence thm, $\exists\ \pi_{x_0} : \Omega \longrightarrow M = \partial(j_{x_0}(X))$

<u>Set</u>: $(\pi_{x_0})_* P_{x_0} := \nu_{x_0}$

Let $D_n = B_d(x_0, n)$, $\tau_n((z)) = \sup\{k \mid z_k \in D_n\}$, $n \leq \tau_n$ and a.s. $\tau_n < \infty \Rightarrow \tau_n \longrightarrow \infty$

$z_{\tau_n} \longrightarrow \pi(z)$

hence $\frac{g(\cancel{z}_{\tau_n}, x)}{g(z_{\tau_n}, x_0)} = ev_x[j_{x_0}(z_{\tau_n})] \longrightarrow ev_x[j_{x_0}(\pi(z))] = \frac{g(\pi(z), x)}{g(\pi(z), x_0)}$

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for all $y$:

$$\int_M \frac{g(m,x)}{g(m,x_0)}\, d\mu_{x_0}(m) = \int_\Omega \frac{g(\pi(z),x)}{g(\pi(z),x_0)}\, dP_{x_0}((z)) \longleftarrow \int_\Omega \frac{g(z_{\tau_n},x)}{g(z_{\tau_n},x_0)}\, dP_{x_0}((z)) = \int_\Omega \sum_y \delta(z_{\tau_n}, y)\frac{g(y,x)}{g(y,x_0)}\, dP_{x_0}((z))$$

$$= \sum_y \frac{g(y,x)}{g(y,x_0)} P_{x_0}(z_{\tau_n} = y) = \sum_y P_x(z_{\tau_n} = y) = 1$$

<u>Cor</u>: $\mu_{x_0}$ represents $1$ on $M$.