Math 205
Integration and calculus of several variables

week 1 - March 30, 2009

1. Introduction

The purpose of this course is to study integration of functions of several variables. I assume you have all been exposed to differential and integral calculus in one variable and to the derivative (jacobian) for functions of several variables. (Of course, these notions will be recalled as needed.) The real numbers are denoted $\mathbb{R}$. Numbers like

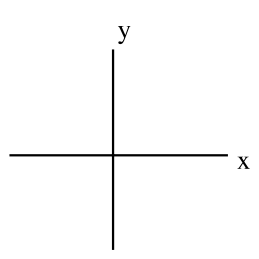
$$1,\ -1,\ 14.23,\ \pi = 3.14159\ldots$$

are real numbers. You may have encountered a more precise definition in terms of Cauchy sequences of rational numbers, but we won't focus on this sort of thing. (I bet you forget it anyway.) Instead we will be concerned with higher dimensions, e.g. $\mathbb{R}^2,\ \mathbb{R}^3,\ \mathbb{R}^4, \ldots,\ \mathbb{R}^n$. Algebraically, we may write (the notation := means "defined to be")

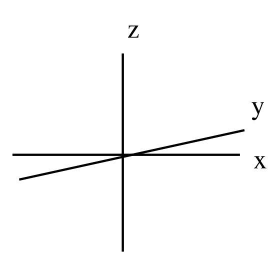
$$\begin{aligned}
\mathbb{R}^2 &:= \{(x, y) \mid x, y \in \mathbb{R}\} \\
\mathbb{R}^3 &:= \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \\
\mathbb{R}^4 &:= \{(x, y, z, t) \mid x, y, z, t \in \mathbb{R}\} \\
\mathbb{R}^5 &:= \{(x, y, z, t, s) \mid x, y, z, t, s \in \mathbb{R}\} \\
&\vdots \\
\mathbb{R}^n &:= \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \le i \le n\}
\end{aligned}$$

In physics, for example, the fourth dimension t is frequently taken to be time and s can be thought of as short for scotch, (the fifth dimension, get it?).

$\mathbb{R}^2$



$\mathbb{R}^3$



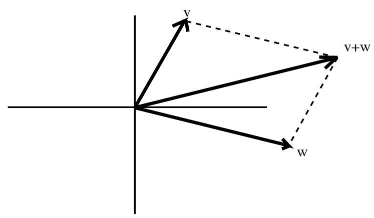
The spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ are *vector spaces*. Algebraically, this means that elements (*vectors*) can be added, subtracted, or multiplied by *scalars* (elements of $\mathbb{R}$). For example

$$-5(1.2, 3.4) = (-6, -17); \quad (1.2, 3.4, 5.6) - (.2, .3, -4) = (1, 3.1, 9.6)$$

or, more generally for $a, b \in \mathbb{R}$

$$a(x_1, \ldots, x_n) + b(y_1, \ldots, y_n) = (ax_1 + by_1, \ldots, ax_n + by_n).$$

In low dimensions, one can visualize this operation geometrically:



Exercise 1. *Justify this picture. I.e., associate to a vector $v = (v_1, v_2) \in \mathbb{R}^2$ the arrow with tail at the origin and head at v. Show the vector $(v_1 + w_1, v_2 + w_2)$ is obtained by parallel translating the arrow $w = (w_1, w_2)$ so its tail is at (v_1, v_2).*

There is a nice notion of *distance* between points p and q in $\mathbb{R}^m$. Suppose $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$. The distance $d(p, q)$ is defined by

$$d(p,q) = (\sum_{i=1}^{m} (p_i - q_i)^2)^{\frac{1}{2}}. \tag{1}$$

For example, in $\mathbb{R}^2$,

$$d((1,2), (-3,4)) = ((1 - (-3))^2 + (2 - 4)^2)^{\frac{1}{2}} = 2\sqrt{5}$$

Exercise 2. *Verify the following properties of $d(p, q)$.*

a. $d(p, q) = d(q, p)$.
b. $d(p, q) \geq 0$. $d(p, q) = 0$ *if and only if* $p = q$.
c. $d(p, q) = d(p - q, 0)$.
d. *If* $m = 1$ *then* $d(p, q) = |p - q|$.

Exercise 3. *Show using the Pythagorean theorem in $\mathbb{R}^2$ that $d(p, q)$ is the length of the vector $p - q$. Use this idea in higher dimension to define the length of a vector $v = (v_1, \ldots, v_n)$.*

We will be interested in functions of several variables. Most typically (but not always!) we will deal with functions $f(x,y)$ or $f(x,y,z)$ with values in $\mathbb{R}$. For example

$$f(x,y) = x+y;\ f(x,y,z) = \exp(xyz);\ f(z_1,\ldots,z_n) = a_1z_1+\ldots+a_nz_n$$

These guys are complicated enough, but when we talk about change of variables, we will also see functions $F : \mathbb{R}^n \to \mathbb{R}^n$, e.g.

$$F(x,y) = (\cos(x), \sin(x))$$
$$G(x_1,x_2) = (7x_1 + 5x_2, -x_1 + .3x_2).$$

More general still are functions $H : \mathbb{R}^m \to \mathbb{R}^n$. A very important example of such a function is a *linear transformation*

$$(2)\quad H(x_1,\ldots,x_m) =$$
$$(a_{11}x_1 + \ldots + a_{1m}x_m, a_{21}x_1 + \ldots + a_{2m}x_m, \ldots, a_{n1}x_1 + \ldots + a_{nm}x_m)$$

where the $a_{ij} \in \mathbb{R}$. For example, here are some linear transformations

$$H(x_1,x_2) = x_1 - x_2;\ H(x,y) = (x,y,x+y); H(x,y,z) = (x,z)$$
$$H(x) = -3.2x;\ H(x_1,\ldots,x_n) = (x_1, x_1+x_2, \ldots, x_1+x_2+\ldots+x_p).$$

Exercise 4. *An $n \times m$ matrix A is a rectangular array of numbers*

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{12} & \ldots & a_{nm} \end{pmatrix}.$$

The linear transformation H in (2) above is obtained by applying A to the column vector of x's

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{12} & \ldots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1m}x_m \\ a_{21}x_1 + \ldots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + \ldots + a_{nm}x_m \end{pmatrix}.$$

Write down the linear transformations associated to the matrices

$$\begin{pmatrix} 4 & 7 \\ -1 & 0 \\ 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$

Write down the matrices corresponding to the linear transformations

$$H(x) = 3x;\ H(x_1,x_2) = (x_1+x_2, x_1-x_2); H(x_1,x_2,x_3) = x_1 - x_2 - x_3$$

A function $\psi : \mathbb{R} \to \mathbb{R}^n$ is called a *path*. Paths are fun because you can try to draw them. For example,

$$\psi(t) = (t, t^2, t^3);\ \ \psi(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Exercise 5. *Sketch the image in* $\mathbb{R}^3$ *(respectively* $\mathbb{R}^2$*) of the interval* $[0, 5]$ *(resp.* $[0, 1]$*) under the paths* ψ *above.*

Many times a function will only be defined on some *domain of definition* $V \subset \mathbb{R}^m$. In symbols, we have

$$\mathbb{R}^m \supset V \xrightarrow{F} \mathbb{R}^n$$

For example, $F(x, y) = x/y$ has domain of definition

$$V = \mathbb{R}^2 - \{(x, 0) \mid x \in \mathbb{R}\}.$$

Functions of several variables are complicated. Attached are Mathematica printouts of two ways of thinking about the function $F(x, y) = \sin(xy)$ in the range $0 \leq x, y \leq 3$. The first, labeled "graph of Sin[xy]" is just the graph, i.e. the locus of all points $(x, y, \sin(xy))$ in $\mathbb{R}^3$. The second, labeled "contour of Sin[xy]" sketches a set of "level lines" $\sin(xy) = \text{constant}$ in $\mathbb{R}^2$.

Exercise 6. *Draw the graph and a contour diagram for the function* $f(x, y) = xy$ *over the square* $0 \leq x, y \leq 2$.

One final concept in this section is the notion of *continuity* for a function $f : \mathbb{R}^m \to \mathbb{R}^n$. The function f is continuous at $p \in \mathbb{R}^m$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(p, q) < \delta \Rightarrow d(f(p), f(q)) < \epsilon$. When $n = 1$ this reads $d(p, q) < \delta \Rightarrow |f(p) - f(q)| < \epsilon$.

Exercise 7. *Show the linear transformation* H *from* (2) *is continuous at every point* p.