Math 205
Integration and calculus of several variables

week 7 - May 11, 2009

10. d is for Disaster

We have two formulas for d.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \tag{10.1}$$

$$d(fdx + gdy) = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx \wedge dy \tag{10.2}$$

Recall also, we were led to impose relations

$$dx \wedge dy = -dy \wedge dx; \qquad dx \wedge dx = 0.$$

These came because in computing integrals, the order of the d's told you the order of the rows in the jacobian determinant, and switching rows changed the sign of the determinant.

We can rewrite (10.2) above as

$$d(fdx + gdy) = (df) \wedge dx + (dg) \wedge dy. \tag{10.3}$$

To check (10.3), we compute

$$\begin{aligned} (df) \wedge dx + (dg) \wedge dy &= (\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy) \wedge dx + (\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy) \wedge dy \\ &= \frac{\partial f}{\partial x} dx \wedge dx + \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial g}{\partial y} dy \wedge dy \\ &= (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx \wedge dy. \end{aligned}$$

Formula (10.3) suggests a definition of $d\omega$ for any differential form ω. Here is the general formula for a p-form on $\mathbb{R}^n$.

Definition 1.

$$\begin{aligned} &d\Big(\sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} f_{i_1,\ldots,i_p}(x_1, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \Big) \\ &\qquad = \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} df_{i_1,\ldots,i_p}(x_1, \ldots, x_n) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \end{aligned}$$

Example 2. *Let* $\omega = fdx + gdy + hdz$ *be a 1-form on* $\mathbb{R}^3$*. Then*

$$
\begin{aligned}
(10.4)\quad d\omega &= df\wedge dx + dg\wedge dy + dh\wedge dz \\
&= (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz)\wedge dx + (\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz)\wedge dy \\
&\quad + (\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz)\wedge dz \\
&= (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dx\wedge dy + (\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z})dx\wedge dz + (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})dy\wedge dz.
\end{aligned}
$$

Example 3. *Let* $\eta = fdx\wedge dy + gdx\wedge dz + hdy\wedge dz$ *be a 2-form in* $\mathbb{R}^3$*. Then*

$$
\begin{aligned}
(10.5)\quad d\eta &= df\wedge dx\wedge dy + dg\wedge dx\wedge dz + dh\wedge dy\wedge dz \\
&= (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz)\wedge dx\wedge dy + (\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz)\wedge dx\wedge dz \\
&\quad + (\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz)\wedge dy\wedge dz \\
&= \frac{\partial f}{\partial z}dz\wedge dx\wedge dy + \frac{\partial g}{\partial y}dy\wedge dx\wedge dz + \frac{\partial h}{\partial x}dx\wedge dy\wedge dz \\
&= \frac{\partial f}{\partial z}dx\wedge dy\wedge dz - \frac{\partial g}{\partial y}dx\wedge dy\wedge dz + \frac{\partial h}{\partial x}dx\wedge dy\wedge dz \\
&= \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right)dx\wedge dy\wedge dz.
\end{aligned}
$$

Notice here that whenever we interchange 2 d*'s, we switch signs. Thus*

$$dz\wedge dy\wedge dx = -dx\wedge dy\wedge dz; \quad dz\wedge dx\wedge dy = -dx\wedge dz\wedge dy = +dx\wedge dy\wedge dz.$$

Here are some properties of d:

Proposition 4. (i) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
(ii) $d(d\omega) = 0$.

Proof. (i) is easy to check. Using it, and renumbering the coordinates if necessary, we reduce (ii) to showing

$$d(d(f(x_1, \ldots, x_n)dx_1\wedge dx_2\wedge \ldots \wedge dx_p)) = 0.$$

We compute

$$
\begin{aligned}
&d(d(f(x_1,\dots,x_n)dx_1\wedge dx_2\wedge\dots\wedge dx_p))\\
&= d\Big(\sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i\Big)\wedge dx_1\wedge\dots\wedge dx_p = \sum_{i=1}^n d\Big(\frac{\partial f}{\partial x_i}\Big)\wedge dx_i\wedge dx_1\wedge\dots\wedge dx_p\\
&\qquad\qquad = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j}dx_j\wedge dx_i\wedge dx_1\wedge\dots\wedge dx_p.
\end{aligned}
$$

Notice in this formula, if $i\neq j$ the second partial $\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{\partial^2 f}{\partial x_j\partial x_i}$ appears twice, once multiplying $dx_i\wedge dx_j\wedge\dots$ and again multiplying $dx_j\wedge dx_i\dots$. These two terms cancel. Also, $\frac{\partial^2 f}{\partial^2 x_i}$ appears multiplying $dx_i\wedge dx_i\wedge\dots$, so these terms die as well. We conclude $d^2(\omega)=0$ as claimed. ☐

Given a p-form ω and a q-form η, we may form their *wedge product*, $\omega\wedge\eta$.

Example 5. *The product of* 1*-forms in* $\mathbb{R}^2$ *looks like*

$$(fdx+gdy)\wedge(Fdx+Gdy)=(fG-Fg)dx\wedge dy.$$

Even simpler, the product of a 0*-form and a* 1*-form in* $\mathbb{R}^2$ *is*

$$f\cdot(Fdx+Gdy)=fFdx+fGdy.$$

(Of course, the product of 0*-forms* f *and* g *is just their product as functions* fg*.) Finally, a typical product of* 2*-forms is*

$$
\begin{aligned}
(fdx_1\wedge dx_3)\wedge(gdx_2\wedge dx_4) &= fgdx_1\wedge dx_3\wedge dx_2\wedge dx_4\\
&= -fgdx_1\wedge dx_2\wedge dx_3\wedge dx_4.
\end{aligned}
$$

Proposition 6. $d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^p\omega\wedge d\eta.$

Proof. Suppose first that $\omega=f$ and $\eta=g$ are 0-forms, i.e. functions. Then

$$d(fg)=\sum\frac{\partial(fg)}{\partial x_i}dx_i=\Big(\sum\frac{\partial f}{\partial x_i}dx_i\Big)g+f\sum\frac{\partial g}{\partial x_i}dx_i=(df)\cdot g+f\cdot dg,$$

proving the proposition in this case. More generally, using linearity, we reduce the proof to the case

$$\omega=fdx_{i_1}\wedge\dots\wedge dx_{i_p}\qquad \eta=gdx_{j_1}\wedge\dots\wedge dx_{j_q}$$

(In other words, the general ω and η are sums of such forms, and one sees that both sides of the desired equality involve the same sums...)

To shorten, let me write dx_I in place of $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and similarly $dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_q}$. Then

$$d\omega = df \wedge dx_I; \qquad d\eta = dg \wedge dx_J$$

$$\begin{aligned} d(\omega \wedge \eta) = d(fg) \wedge dx_I \wedge dx_J &= f(dg) \wedge dx_I \wedge dx_J + g(df) \wedge dx_I \wedge dx_J \\ &= (df \wedge dx_I) \wedge (g dx_J) + (-1)^p (f dx_I) \wedge (dg \wedge dx_J) \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge d\eta. \end{aligned}$$

□

Exercise 7.
1. *Notice we have used in the proof of proposition 6 that*
$$dg \wedge dx_I = (-1)^p dx_I \wedge dg.$$
Verify this in detail.
2. *Prove $d(f dg_1 \wedge dg_2 \wedge \ldots \wedge dg_p) = df \wedge dg_1 \wedge \ldots \wedge dg_p$. (Hint: use induction on p, i.e. first consider the case $p = 0$. Then show if the assertion is true for p, it is true for $p+1$.)*

Of course, this is mere symbol manipulation; worthless unless it gives us some insight into the process of integration. What we are heading for is a proof of Stokes' formula

$$\int_\phi d\omega = \int_{\partial\phi} \omega$$

in a more general framework. First however, I want to change slightly our viewpoint. Let $\phi : I^p \to U \overset{\text{open}}{\subset} \mathbb{R}^n$, and let ω be a p-form on U. I want to define a p-form $\phi^*\omega$ on I^p in such a way that

$$\int_\phi \omega = \int_{I^p} \phi^*\omega. \tag{10.6}$$

In other words, we need a mapping

$$\phi^* : \{p - \text{forms on } U\} \to \{p - \text{forms on } I^p\} \tag{10.7}$$

such that (10.6) holds. In fact, we have a mapping ϕ^* defined for forms of any degree.

Proposition 8. *Let ϕ be as above. For $q \geq 0$ There exists a unique mapping*

$$\phi^* : \{q - \textit{forms on } U\} \to \{q - \textit{forms on } I^p\}$$

satisfying the following conditions

(i) *On 0-forms (=functions), $\phi^*(f) := f \circ \phi$.*
(ii) *$\phi^*(d\omega) = d(\phi^*\omega)$.*
(iii) *ϕ^* respects sums, i.e. $\phi^*(\omega_1 + \omega_2) = \phi^*(\omega_1) + \phi^*(\omega_2)$.*
(iv) *ϕ^* respects products, i.e. $\phi^*(\omega_1 \wedge \omega_2) = \phi^*(\omega_1) \wedge \phi^*(\omega_2)$.*

Proof. What we will do is use the conditions of the proposition to get an expression for what should be $\phi^*\omega$. Then we will check that this expression may be used as definition. Let $x_1, \ldots, x_n$ be coordinates on U, and let $t_1, \ldots, t_p$ be coordinates on I^p. Write $\phi(t) = (x_1(t), \ldots, x_n(t))$. Apply (ii) to the 0-form x_i on U to get

$$\phi^*(dx_i) = d(\phi^*(x_i)) = d(x_i(t_1, \ldots, t_p)) = \sum_{j=1}^{p} \frac{\partial x_i}{\partial t_j} dt_j.$$

Now apply compatibility with multiplication (iv) above to deduce

$$\begin{aligned} &\phi^*(f_{i_1,\ldots,i_q} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_q}) \\ &= f_{i_1,\ldots,i_q}(x_1(t), \ldots, x_n(t))(\sum_{j=1}^{p} \frac{\partial x_{i_1}}{\partial t_j} dt_j) \wedge (\sum_{j=1}^{p} \frac{\partial x_{i_2}}{\partial t_j} dt_j) \wedge \\ &\qquad \ldots \wedge (\sum_{j=1}^{p} \frac{\partial x_{i_q}}{\partial t_j} dt_j). \end{aligned} \tag{10.8}$$

The most general ω is given by a sum of such terms as in definition 1. Using compatibility with sums (iii) above, we deduce a formula for $\phi^*\omega$ as a sum of expressions like the right hand side of (10.8) above. The general formula is a bit painful to write out, but it is clear that such a formula exists, defining $\phi^*\omega$.

We're not quite done, because we should check that the ϕ^* we have defined in (10.8) satisfies the desired properties (i)-(iv). (Just because we used these properties to define ϕ^* doesn't mean that they will hold in general.) The only condition that is tricky is (ii). Suppose first the $\omega = f$ is a function. Then

$$\begin{aligned} d(\phi^*(f)) = d(f \circ \phi) &= \sum_j \frac{\partial (f \circ \phi)}{\partial dt_j} \\ &= \sum_i \frac{\partial f}{\partial x_i}(\phi(t)) \sum_j \frac{\partial x_i(t)}{\partial t_j} dt_j = \phi^*(df), \end{aligned}$$

which proves (ii) in this case. (Notice this is just our old friend the chain rule.) Next assume $p > 0$. Renumbering the coordinates and using linearity as before, we can check (ii) for $\omega = f dx_1 \wedge \ldots \wedge dx_p$. We have (using exercise 7 and (iv))

$$\begin{aligned} d(\phi^*\omega) &= d(\phi^*(f dx_1 \wedge \ldots \wedge dx_p)) \\ &= d\Big(f(x(t)) d(x_1(t)) \wedge \ldots \wedge d(x_p(t))\Big) \\ &= d(f(x(t))) \wedge d(x_1(t)) \wedge \ldots \wedge d(x_p(t)) \\ &= d(\phi^*(f)) \wedge d(\phi^*(x_1)) \wedge \ldots \wedge d(\phi^*(x_p)) \\ &= \phi^*(df) \wedge \phi^*(dx_1) \wedge \ldots \wedge \phi^*(dx_p) \\ &= \phi^*\Big(df \wedge dx_1 \wedge \ldots \wedge dx_p\Big) = \phi^*(d\omega). \end{aligned}$$

□

Exercise 9. *Let $\phi : I^2 \to \mathbb{R}^n$ be a differentiable map. Show*

$$\phi^*(dx_i \wedge dx_j) = \det \begin{pmatrix} \frac{\partial x_i}{\partial t_1} & \frac{\partial x_i}{\partial t_2} \\ \frac{\partial x_j}{\partial t_1} & \frac{\partial x_j}{\partial t_2} \end{pmatrix} dt_1 \wedge dt_2.$$

Prove a similar result for $\phi^*(dx_i \wedge dx_j \wedge dx_k)$ when $\phi : I^3 \to \mathbb{R}^n$.

Proposition 10. *Let $\phi : I^p \to U \subset \mathbb{R}^n$ be as above, and assume $0 \le p \le 3$. Let ω be a p-form on U. Then*

$$\int_{I^p} \phi^*\omega = \int_\phi \omega$$

Proof. In fact, this is an easy consequence of exercise 9. Suppose, for example, just to be nasty that $p = 3$. The linearity trick we have used before reduces us to the case $\omega = f dx_i \wedge dx_j \wedge dx_k$. We have from the exercise

$$\begin{aligned} \phi^*\omega &= f(x(t))\phi^*(dx_i \wedge dx_j \wedge dx_k) \\ &= f(x(t)) \det \begin{pmatrix} \frac{\partial x_i}{\partial t_1} & \frac{\partial x_i}{\partial t_2} & \frac{\partial x_i}{\partial t_3} \\ \frac{\partial x_j}{\partial t_1} & \frac{\partial x_j}{\partial t_2} & \frac{\partial x_j}{\partial t_3} \\ \frac{\partial x_k}{\partial t_1} & \frac{\partial x_k}{\partial t_2} & \frac{\partial x_k}{\partial t_3} \end{pmatrix} dt_1 \wedge dt_2 \wedge dt_3. \end{aligned}$$

On the other hand, *by definition*

$$\int_\phi \omega = \int_{I^3} f(x(t)) \det \begin{pmatrix} \frac{\partial x_i}{\partial t_1} & \frac{\partial x_i}{\partial t_2} & \frac{\partial x_i}{\partial t_3} \\ \frac{\partial x_j}{\partial t_1} & \frac{\partial x_j}{\partial t_2} & \frac{\partial x_j}{\partial t_3} \\ \frac{\partial x_k}{\partial t_1} & \frac{\partial x_k}{\partial t_2} & \frac{\partial x_k}{\partial t_3} \end{pmatrix} dt_1 \wedge dt_2 \wedge dt_3.$$

This is the same as the integral over I^3 of $\phi^*\omega$ as calculated above. □

We can now prove Stokes' Theorem for 2-forms.

Theorem 11. *Let $\phi : I^2 \to U \overset{open}{\subset} \mathbb{R}^n$ be differentiable, and Let ω be a 1-form on U. Then*

$$\iint_\phi d\omega = \int_{\partial\phi} \omega.$$

Proof. All the tools we need are now at hand. Namely

$$\iint_\phi d\omega = \iint_{I^2} \phi^*(d\omega) = \iint_{I^2} d(\phi^*\omega) \overset{(*)}{=} \int_{\partial I^2} \phi^*\omega = \int_{\partial\phi} \omega.$$

□

We have used here the fact that we already checked Stokes theorem on I^2. Once we do the same on I^3, Stokes' theorem for 3-forms will be equally easy. To begin, we invoke the great Fubini:

$$\iiint_{I^3} \frac{\partial f}{\partial t_i} dt_1 \wedge dt_2 \wedge dt_3 = \int_{t_1=0}^{1} dt_1 \dots \int_{t_i=0}^{1} \frac{\partial f}{\partial t_i} dt_i \dots \int_{t_3=0}^{1} dt_3 \tag{10.9}$$

$$= \iint_{I^2} (f(t_1, \dots, 1, \dots, t_3) - f(t_1, \dots, 0, \dots, t_3)) dt_1 \dots \hat{dt_i} \dots dt_3.$$

The hat ˆ here means to leave out that differential, and the 0 and 1 occur in the i-th place.

Definition 12.

$$\partial I^3 :=$$

$$\Big((t_1, t_2, 1) - (t_1, t_2, 0)\Big) - \Big((t_1, 1, t_3) - (t_1, 0, t_3)\Big) + \Big((1, t_2, t_3) - (0, t_2, t_3)\Big)$$

$$= (t_1, t_2, 1) - (t_1, t_2, 0) - (t_1, 1, t_3) + (t_1, 0, t_3) + (1, t_2, t_3) - (0, t_2, t_3).$$

Here e.g. $(t_1, t_2, 1)$ means the 2-chain $I^2 \to I^3$; $(t_1, t_2) \mapsto (t_1, t_2, 1)$. We always orient these 2-chains by ordering the coordinates t_i, t_j with $i < j$.

Proposition 13. *Let ω be a 2-form on I^3. Then $\iiint_{I^3} d\omega = \iint_{\partial I^3} \omega$.*

Proof. Write $\omega = f dt_1 \wedge dt_2 + g dt_1 \wedge dt_3 + h dt_2 \wedge dt_3$. Compute

$$d\omega = \Big(\frac{\partial f}{\partial t_3} - \frac{\partial g}{\partial t_2} + \frac{\partial h}{\partial t_1}\Big) dt_1 \wedge dt_2 \wedge dt_3$$

Now it is just a matter of referring back to (10.9) to compute the integral over I^3 of $d\omega$. □

Theorem 14 (Stokes' Theorem for 3-forms)**.** *Let $\phi : I^3 \to \mathbb{R}^n$, and let ω be a 2-form on $\mathbb{R}^n$. Then $\iiint_\phi d\omega = \iint_{\partial\phi} \omega$.*

Exercise 15. *Prove theorem 14.*

Exercise 16. *Compute $d\omega$ in the following examples*

$$\omega = e^x dy \wedge dz; \quad \omega = dx/y; \quad \omega = xdy \wedge dz + ydx \wedge dz + zdx \wedge dy$$

Exercise 17. *Compute $\phi^*\omega$ in the following examples*

$$\phi(t) = (t,t),\ \omega = \log x dy - e^y dx; \quad \phi(t_1,t_2) = (t_1,t_2,t_1t_2),\ \omega = xdy \wedge dz.$$