

Math 205

Integration and calculus of several variables

week 9 - May 27, 2009

12. DIV, GRAD, CURL

We introduced vector fields as an alternative way to think about differential forms and integration. The down side was that one needs lengths and angles, which are not strictly speaking necessary to integrate differential forms. On the other hand, one sees beautiful tie-ins with physics (fluid flow, Maxwell's equations). One can also speak of arc length and surface area in \mathbb{R}^n which are not defined just using differential forms. Here is a summary of the various terms.

$$\nabla := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

Grad in \mathbb{R}^n

$$\text{Grad}(f) := \nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right); \quad \int_{\phi} df = \int_a^b \nabla f \cdot \frac{d\vec{\phi}}{dt} dt$$

Div in \mathbb{R}^2

$$\text{Div}(f, g) := \nabla \cdot (f, g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}; \quad \int_{\varphi(I^2)} \text{Div}(f, g) dx \wedge dy = \int_{\partial\varphi} (f, g) \cdot \vec{n} ds$$

Div in \mathbb{R}^3

$$\begin{aligned} \text{Div}(f_1, f_2, f_3) &:= \nabla \cdot (f_1, f_2, f_3) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}; \\ \omega &:= f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2 \\ d\omega &= \text{Div}(f_1, f_2, f_3) dx_1 \wedge dx_2 \wedge dx_3 \\ \int_{\varphi(I^2)} \text{Div}(f_1, f_2, f_3) dx_1 \wedge dx_2 \wedge dx_3 &= \int_{\partial\varphi} \omega = \int_{\partial\varphi} (f_1, f_2, f_3) \cdot \vec{n} d\sigma \end{aligned}$$

Curl in \mathbb{R}^2

$$\begin{aligned} \text{Curl}(f, g) &= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}; & d(fdx + gdy) &= \text{Curl}(f, g)dx \wedge dy \\ \int_{\varphi(I^2)} \text{Curl}(f, g)dx \wedge dy &= \int_{\partial\varphi} (f, g) \cdot \vec{T} ds \end{aligned}$$

Curl in \mathbb{R}^3

$$\begin{aligned} \text{Curl}(f_1, f_2, f_3) &:= \nabla \times (f_1, f_2, f_3) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \\ \omega &= f_1 dx_1 + f_2 dx_2 + f_3 dx_3; \\ \int_{\varphi(I^2)} d\omega &= \int_{\varphi(I^2)} \text{Curl}(f_1, f_2, f_3) \cdot \vec{n} d\sigma = \int_{\partial\varphi} \omega = \int_{\partial\varphi} (f_1, f_2, f_3) \cdot \vec{T} ds \end{aligned}$$

13. APPLICATIONS

We discussed two applications.

I. **Fluid flow.** Let $\vec{v} = (f, g)$ be a vector field in \mathbb{R}^2 . We imagine a fluid with constant surface density flowing steadily over a region in \mathbb{R}^2 . At any point (x_0, y_0) the vector field $v(x_0, y_0) = (f(x_0, y_0), g(x_0, y_0))$ represents the velocity of the fluid at that point. “Steady” flow means that this velocity does not change with time. if we imagine the region parametrized by $\varphi : I^2 \rightarrow \mathbb{R}^2$, then

$$\int_{\varphi} \text{Curl} \vec{v} dx \wedge dy = \int_{\partial\varphi} \vec{v} \cdot \vec{T} ds$$

measures the rotation of the fluid, while

$$\int_{\varphi} \text{Div}(\vec{v}) dx \wedge dy = \int_{\partial\varphi} \vec{v} \cdot \vec{n} ds$$

measures the total amount of fluid leaving the region. (One has, here, to be careful with the signs.)

II. **Maxwell’s equations in electromagnetism.** To simplify (I did a more complicated case in class) let us consider the case of a vacuum where charge and current densities are zero. We assume further that the electric field $\vec{E} = (e_1, e_2, e_3)$ and the magnetic field $\vec{B} = (b_1, b_2, b_3)$ (these are vector fields in \mathbb{R}^3) do not vary with time. then Maxwell’s equations say:

$$\begin{aligned}\nabla \cdot \vec{B} &= 0 = \nabla \times \vec{B} \\ \nabla \cdot \vec{E} &= 0 = \nabla \times \vec{E}.\end{aligned}$$

Define 1-forms and 2-forms

$$\begin{aligned}\omega_B &= b_1 dx_1 + b_2 dx_2 + b_3 dx_3; & \omega_E &= e_1 dx_1 + e_2 dx_2 + e_3 dx_3; \\ \eta_B &= b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2; \\ \eta_E &= e_1 dx_2 \wedge dx_3 + e_2 dx_3 \wedge dx_1 + e_3 dx_1 \wedge dx_2.\end{aligned}$$

Then Maxwell's equations in this simple case translate to say that these differential forms are all *closed*. That is

$$d\omega_B = d\omega_E = 0 = d\eta_B = d\eta_E.$$

14. CLOSURE

It turns out to be very important that $d \circ d = 0$. For example $d(d(xyz)) = d(xdy \wedge dz + ydx \wedge dz) = dx \wedge dy \wedge dz + dy \wedge dx \wedge dz = 0$ because $dx \wedge dy + dy \wedge dx = 0$. Recall the proof:

$$dd\omega = d(d \sum_I f_I dx^I) = d \sum df_I \wedge dx^I = \sum_I (ddf_I) dx^I;$$

so it suffices to show $ddf = 0$ for a function f . But

$$\begin{aligned}ddf &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = \\ &= \sum_{a=1}^n \frac{\partial^2 f}{\partial x_a^2} (dx_a \wedge dx_a) + \sum_{a < b} \frac{\partial^2 f}{\partial x_a \partial x_b} (dx_a \wedge dx_b + dx_b \wedge dx_a) = 0.\end{aligned}$$

Another way to say this is to say in some degree p that

$$\{\text{exact forms of degree } p\} \subset \{\text{closed forms of degree } p\}.$$

In other words, forms $d\omega$ are necessarily closed, i.e. $dd\omega = 0$.

These are vector spaces, so we may form the quotient vector space

$$H^p = \{\text{closed forms of degree } p\} / \{\text{exact forms of degree } p\}.$$

The quotient H^p is called the p -th de Rham cohomology of the domain on which the forms are defined. It is a very important invariant. We won't be able to pursue these ideas further, but let us at least see that H^p is not zero in general.

In fact we have already done this computation. Take D to be an open disk of radius 2 about the origin in \mathbb{R}^2 , and let $D^* = D - \{(0, 0)\}$ be the punctured disk. Consider the 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

on D^* . Check that $d\omega = 0$, i.e. that ω is closed. Check that in fact,

$$\omega = d(\arctan(y/x))$$

At first this might seem to mean that ω is exact, hence zero in $H^1(D^*)$, but the function $\arctan(y/x)$ is multiple-valued on D^* . (Note that $\arctan(y/x)$ is the function θ in polar coordinates.) If there was a well-defined function f on D^* with $df = \omega$, we would get

$$d(f - \arctan(y/x)) = 0.$$

It would follow that $f - \arctan(y/x) = c$ is a constant. But then we would have

$$\arctan(y/x) = f - c,$$

a well-defined function. Thus, there exist closed 1-forms on D^* which are not exact.

Practice Problems for Final Exam

Do these problems in lieu of homework for this week.

One or more of them will appear on the final.

You do not need to hand them in.

1. Define the operators Div , $Grad$, $Curl$. For f a function on \mathbb{R}^2 show $Curl(Grad(f)) = 0$. Interpret this identity in terms of differential forms. For $\vec{v} = (f_1, f_2, f_3)$ a vector field on \mathbb{R}^3 , show $Div(Curl(\vec{v})) = 0$. Interpret this identity in terms of differential forms.
2. Let D^* , ω be as in the text above. Let $S \subset \mathbb{R}^2$ be the circle of radius 1 about the origin. Compute $\int_S \omega$. Show that if ω were exact, then $\int_S \omega = 0$. Conclude that ω is not exact.
3. Recall the boundary of $I = [0, 1]$ is the formal sum $\partial I = (1) - (0)$

and the boundary of $I^2 = [0, 1] \times [0, 1]$ is

$$\begin{aligned} \partial I^2 = \partial I \times I - I \times \partial I = \\ \{(1, u) \mid u \in [0, 1]\} - \{(0, u) \mid u \in [0, 1]\} - \\ \{(t, 1) \mid t \in [0, 1]\} + \{(t, 0) \mid t \in [0, 1]\}. \end{aligned}$$

Let $\varphi : I^2 \rightarrow \mathbb{R}^3$ be $\varphi(\theta, \mu) = (\cos(\pi\theta) \cos(2\pi\mu), \cos(\pi\theta) \sin(2\pi\mu), \sin(\pi\theta))$. Let ω be a 1-form on \mathbb{R}^3 . show

$$\int_{\varphi} d\omega = 0.$$

4. Give a careful definition of the arc length form ds . Use it to compute the arc length of the ellipse $\{(a \cos(2\pi t), b \sin(2\pi t)) \mid 0 \leq t \leq 1\}$. You need not actually compute the integral. Just write the arc length as an integral over the interval $[0, 1]$.

5. Give a careful definition of the surface area form $d\sigma$ on \mathbb{R}^3 . This is more difficult than the previous problem. For full credit, you should start with a definition of the cross product in \mathbb{R}^3 and a discussion of its geometric properties. Then write $d\sigma$ in terms of the cross product of tangent vectors for a parametrized surface $\varphi : I^2 \rightarrow \mathbb{R}^3$. Compute the surface area when $\phi(t, u) = (t, u, 3t - 2u)$. (Again you need not actually evaluate the integral you get.)

6. Sketch the following vector fields

$$\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right); \quad (z, x, y); \quad (0, -z, y)$$

Compute the divergences for the vector fields $(x, 0)$, $(y, 0)$. Explain geometrically why one divergence is zero and the other is not.