Math 205
Integration and calculus of several variables
week 2 - April 6, 2009

2. Multiple Integrals

In this section, we consider the problem of defining the integrals

$$\iint_R f(x_1, x_2)dx_1dx_2; \quad \iiint_R f(x_1, x_2, x_3)dx_1dx_2dx_3; \tag{2.1}$$
$$\int \cdots \int_R f(x_1, \ldots, x_n)dx_1 \cdots dx_n$$

where R is a rectangle $a_1 \le x \le b_1$; $a_2 \le y \le b_2$ (resp. a hyperrectangle defined by $a_i \le x_i \le b_i$). For simplicity we will assume the function f is continuous on R, though in fact later we will need to apply the result to a slightly more general situation where f is a finite sum of functions f_α defined and continuous on suitable subsets $S_\alpha \subset R$ and zero outside S_α. We will formulate an exercise to extend the results to that case. Also, for notational simplicity, we will write the details of the proof in the case of $R \subset \mathbb{R}^2$. The general case is exactly the same.

We begin by defining upper and lower Riemann sums. Let $N \ge 1$ be given, and define $\delta_i = \frac{b_i - a_i}{N}$ for $i = 1, 2$. Define little rectangles

$$R_{pq} = \{(x_1, x_2) \mid a_1 + (p-1)\delta_1 \le x_1 \le a_1 + p\delta_1;$$
$$a_2 + (q-1)\delta_2 \le x_2 \le a_2 + q\delta_2\}; \quad 1 \le p, q \le N$$

The R_{pq} are closed bounded regions, hence *compact*, so the continuous function f assumes a maximum M_{pq} and a minimum m_{pq} on R_{pq}. Since the area of R_{pq} is $\delta_1\delta_2$ we would naturally expect

$$\text{lower sum}(N) := \delta_1\delta_2 \sum_{1 \le p,q \le N} m_{pq} \le \iint_R f dx_1dx_2 \le$$
$$\delta_1\delta_2 \sum_{1 \le p,q \le N} M_{pq} := \text{upper sum}(N).$$

Notice

$$\delta_1\delta_2 \sum_{1 \le p,q \le N} M_{pq} = N^2\delta_1\delta_2(\sum M_{pq})/N^2 = \tag{2.2}$$
$$\text{area of } R \cdot \text{average of } M_{pq}$$

A similar equation holds for the lower sums. If we consider a refinement by taking $N' = rN$ the area of R of course stays the same. Suppose

$M'_{p'q'}$ is the supremum of f on one of the smaller rectangles $R'_{p'q'}$. There are r^2 smaller rectangles $R'_{p'q'}$ contained in one larger one R_{pq}, and each $M'_{p'q'} \leq M_{pq}$ so the average of the $M'_{p'q'}$ is also $\leq M_{pq}$. Summing over p, q we conclude

$$\text{upper sum}(rN) \leq \text{upper sum}(N)$$

By the same token, we get the opposite inequality on the lower sums, so put together

$$\text{lower sum}(N) \leq \text{lower sum}(rN) \leq \text{upper sum}(rN) \leq \text{upper sum}(N)$$

Recall the definition of the factorial of an integer

$$N! := 1 \times 2 \times \cdots \times N$$

We see

$$\begin{aligned} \text{lower sum}(N!) \leq \text{lower sum}((N+1)!) \leq \dots \\ \leq \text{upper sum}((N+1)!) \leq \text{upper sum}(N!) \end{aligned}$$

Recall that a decreasing (respectively an increasing) sequence of numbers which is bounded below (resp. bounded above) has a limit. We have thus two possibilities

$$\underline{\iint_R} f dx_1 dx_2 = \lim_{N\to\infty} \text{lower sum}(N!)$$
$$\overline{\iint_R} f dx_1 dx_2 = \lim_{N\to\infty} \text{upper sum}(N!)$$

To show these two definitions coincide, it suffices to show

Lemma 1. *Given $\epsilon > 0$ there exists N_0 such that $N > N_0$ implies*

$$\textit{upper sum}(N) - \textit{lower sum}(N) < \epsilon$$

Proof. The rectangle R is compact, so the continuous function f is in fact *uniformly continuous* on R. This means that given $\epsilon > 0$ there exists $\delta > 0$ such that for **any** (x_1, x_2), $(x'_1, x'_2) \in R$ we have

$$d((x_1, x_2), (x'_1, x'_2)) < \delta \Rightarrow |f(x_1, x_2) - f(x'_1, x'_2)| < \epsilon/\text{area of } R$$

For N sufficiently large, we see that any two points v, v' in the same small rectangle R_{pq} satisfy $d(v, v') < \delta$, so

$$M_{pq} - m_{pq} < \epsilon/\text{area of } R$$
$$\text{average of } M_{pq} - \text{average of } m_{pq} < \epsilon/\text{area of } R$$
$$\text{upper sum}(N) - \text{lower sum}(N) < \epsilon$$

The assertion of the lemma now follows from (2.2). ∎

Extending these arguments in the obvious way from 2 to n dimensions as remarked above, we have proved:

Theorem 2. *Let R be a hyperrectangle in $\mathbb{R}^n$ defined by $a_i \leq x_i \leq b_i;\ 1 \leq i \leq n$. Let $f : R \to \mathbb{R}$ be a continuous function. Then the common limit*

$$\int \cdots \int_R f dx_1 dx_2 \cdots dx_n := \lim_{N\to\infty} \textit{lower sum}(N) = \lim_{N\to\infty} \textit{upper sum}(N)$$

is defined.

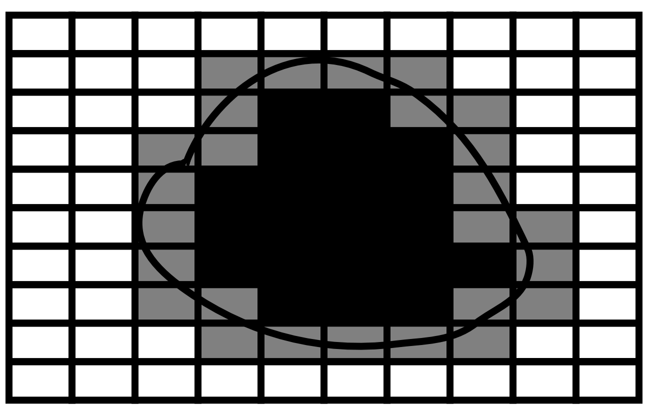
Exercise 3. *Let $S \subset \mathbb{R}^2$ be a closed bounded set, and let $f : S \to \mathbb{R}$ be a continuous function on S. Fix a rectangle $R \subset \mathbb{R}^2$ with $S \subset R$. For $N \geq 1$ given, let $R_{pq} \subset R$ be a decomposition of R into N^2 smaller rectangles as above. Define*

$$\textit{lower volume}(N) = \sum \textit{Volume}(R_{pq}) \quad \textit{sum over all } R_{pq} \subset S$$

$$\textit{upper volume}(N) = \sum \textit{Volume}(R_{pq}) \quad \textit{sum over all } R_{pq} \textit{ meeting } S$$

We say that S has volume V if

$$V = \lim_{N\to\infty} \textit{lower volume}(N) = \lim_{N\to\infty} \textit{upper volume}(N)$$



In the illustration, S is bounded by the curved line. The lower volume is given by the black rectangles, and the upper volume is the sum of the black plus grey rectangles. If the two limits are not equal, we simply say that the volume of S is not defined.

Assume the volume of S is defined, and extend f to a function F on the rectangle R by setting $F = 0$ on $R - S$. Show that even though F

is not necessarily continuous, the same construction as above yields a well defined integral. Use this to define

$$\int \cdots \int_S f dx_1 dx_2 \cdots dx_n := \int \cdots \int_R F dx_1 dx_2 \cdots dx_n$$

Before doing a lot of examples, it seems best to develop the basic technique, sometimes called Fubini's theorem, for calculating such integrals. However, we can cite one basic

Example 4. *Suppose $f : R \to \mathbb{R}$ is the constant function 1. Then $\int \cdots \int_R 1 dx_1 \cdots dx_n =$ Vol.of R.*

3. Fubini's Theorem

To simplify notation, we continue to work with functions of two variables. It will be clear that the results we obtain generalize to functions of n variables.

Theorem 5. *Assume f is continuous on the rectangle R defined by $a_i \leq x_i \leq b_i;\ i = 1, 2$.*

i. *The function $F(x_2) := \int_{a_1}^{b_1} f(x_1, x_2) dx_1$ is defined and continuous on $a_2 \leq x_2 \leq b_2$.*

ii.
$$\iint_R f dx_1 dx_2 = \int_{a_2}^{b_2} F(x_2) dx_2 = \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2 \right)$$

Proof. f is continuous on the closed, bounded set R, so f is uniformly continuous. Given $\epsilon > 0$ there will thus exist $\delta > 0$ such that

$$|x_2' - x_2| < \delta \Rightarrow |f(x_1, x_2') - f(x_1, x_2)| < \frac{\epsilon}{(b_1 - a_1)}.$$

This implies

$$|F(x_1, x_2') - F(x_1, x_2)| = |\int_{a_1}^{b_1} f(x_1, x_2') dx_1 - \int_{a_1}^{b_1} f(x_1, x_2) dx_1|$$
$$\leq \int_{a_1}^{b_1} |f(x_1, x_2') - f(x_1, x_2)| dx_1 \leq \int_{a_1}^{b_1} \frac{\epsilon}{(b_1 - a_1)} dx_1 = \epsilon,$$

so F is continuous.

To prove (ii), again we subdivide R into N^2 smaller rectangles R_{pq}. By uniform continuity, given $\epsilon > 0$ there exists an N such that for $v, v' \in R_{pq}$, $|f(v') - f(v)| < \frac{\epsilon}{(b_1 - a_1)(b_2 - a_2)}$. Let $a_{pq} \in R_{pq}$ be arbitrary points. If x_2 lies in the interval

$$a_2 + (q - 1)(b_2 - a_2)/N \leq x_2 \leq a_2 + q(b_2 - a_2)/N,$$

Riemann sums in 1 variable show

$$|F(x_2) - \frac{(b_1-a_1)}{N}\sum_{p=1}^{N} a_{pq}| =$$

$$|\int_{a_1}^{b_1} f(x_1,x_2)dx_1 - \frac{(b_1-a_1)}{N}\sum_{p=1}^{N} a_{pq}| \le \frac{\epsilon}{(b_2-a_2)}$$

If we now use this to approximate the integral over $F(x_2)$ we get

$$|\int_{a_2}^{b_2} F(x_2)dx_2 - \frac{(b_1-a_1)(b_2-a_2)}{N^2}\sum_{p,q=1}^{N} a_{pq}| \le \epsilon.$$

Finally, we have

$$\text{lower sum}(N) \le \frac{(b_1-a_1)(b_2-a_2)}{N^2}\sum_{p,q=1}^{N} a_{pq} \le \text{upper sum}(N)$$

where lower and upper sums are those used in calculating $\iint_R f(x_1,x_2)dx_1dx_2$. Passing to the limit over N, it follows that

$$\iint_R f(x_1,x_2)dx_1dx_2 = \int_{a_2}^{b_2} F(x_2)dx_2$$

as claimed. □

Example 6. *Suppose we want to calculate* $\int_0^{2\pi}\int_0^{2\pi}\sin(xy)dxdy$. *We first calculate the integral dy treating* x *as a constant*

$$\int_0^{2\pi}\sin(xy)dy = -x^{-1}\cos(xy)|_{y=0}^{2\pi} = x^{-1}(1-\cos(2\pi x)).$$

Next we integrate this result from $x=0$ *to* $x=2\pi$

$$\int_0^{2\pi}\int_0^{2\pi}\sin(xy)dxdy = \int_0^{2\pi} x^{-1}(1-\cos(2\pi x))dx$$

Hmmmm. I seem to have picked an integral which is tricky to evaluate. Well, if I don't err, using the power series expansion for cos *yields*

$$\int_0^{2\pi}\int_0^{2\pi}\sin(xy)dxdy = \frac{(2\pi)^2}{2\cdot 2!} - \frac{(2\pi)^4}{4\cdot 4!} + \frac{(2\pi)^6}{6\cdot 6!} - \dots$$

You might check this, however.

Example 7. *Consider the 3 dim. integral*

$$\iiint_R (x+y+z)dxdydz$$

where $R : 0 \leq x \leq 1;\ -1 \leq y \leq 1; 2 \leq z \leq 4$. *The above discussion leads to the iterated integral*

$$\int_0^1 \left(\int_{-1}^1 \left(\int_2^4 (x + y + z) dz \right) dy \right) dx = \int_0^1 \left(\int_{-1}^1 \left(xz + yz + \frac{z^2}{2} \right) \Bigg|_{z=2}^4 dy \right) dx$$

$$= \int_0^1 \left(\int_{-1}^1 (2x + 2y + 6) dy \right) dx = \int_0^1 (2xy + y^2 + 6y)\big|_{y=-1}^1 dx$$

$$= \int_0^1 (4x + 12) dx = (2x^2 + 12x)\big|_0^1 = 14.$$