Math 205
Integration and calculus of several variables

week 2 - April 6, 2009

3. Examples of multiple integrals

Exercise 1. i. *Calculate*

$$\iint_R (x^2+y^2)dxdy$$

where $R: 0 \leq x \leq 1;\ 0 \leq y \leq 1.$

ii. *Calculate*

$$\iint_R (x^2+y^2)^{\frac{1}{2}}dxdy$$

over the same rectangle as in (i).

iii. *Calculate*

$$\iiint_R \sin(x)\sin(y)\sin(z)dxdydz$$

over the rectangle $R: 0 \leq x \leq \pi;\ 0 \leq y \leq \pi;\ 0 \leq z \leq \pi/2.$

iv. *Give a general formula for the integral*

$$\int \cdots \int_R f_1(x_1)f_2(x_2)\cdots f_n(x_n)dx_1dx_2\dots dx_n$$

over the rectangle $R: a_i \leq x_i \leq b_i;\ 1 \leq i \leq n.$

We now want to modify Fubini's theorem to calculate some integrals over more general regions than rectangles. Let S be a region contained in a rectangle R as in the following illustration.

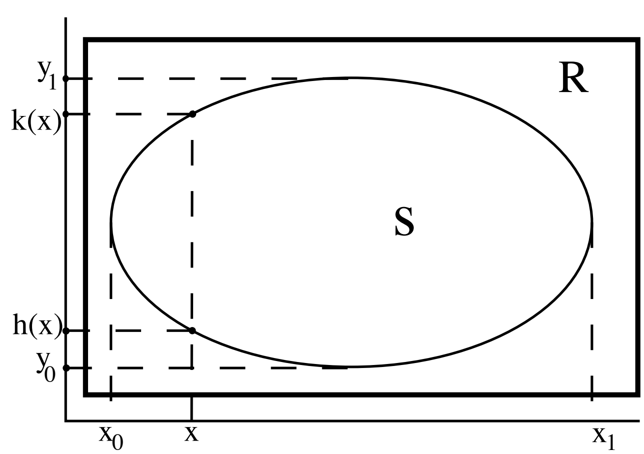

fig. 4

Assume the volume of S is defined as in exercise 3 from section 2. Let $f : S \to \mathbb{R}$ be a continuous function and extend f to a function (again not necessarily continuous) $F : R \to \mathbb{R}$ by setting $F(v) = 0$ for $v \notin S$. By the exercise just cited we have defined $\iint_R F(x, y)dxdy$.

Exercise 2. *With notation as above, prove Fubini's theorem. Show, that is, that*

$$\iint_S f(x,y)dxdy := \iint_R F(x,y)dxdy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} F(x,y)dy \right) dx.$$

where $R : a_1 \leq x \leq b_1;\ a_2 \leq y \leq b_2$.

Notice that the above formula can be rewritten (with reference to fig. 4)

$$\iint_S f(x,y)dxdy = \int_{x_0}^{x_1} \left(\int_{h(x)}^{k(x)} F(x,y)dy \right) dx. \tag{3.1}$$

Here $h(x)$ (respectively $k(x)$) parametrizes the lower (resp. upper) boundary of S.

Example 3. *Suppose we want to compute the integral of the function* $f(x, y) = x - y$ *over the triangle* T *with vertices* $(0, 0)$, $(2, 0)$, *and* $(2, 1)$. *(See illustration)*

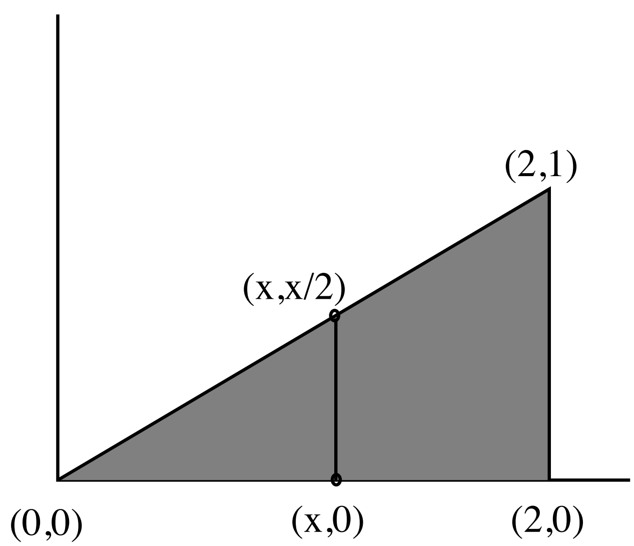

fig. 5

Fubini's theorem yields

$$\iint_T (x-y)dxdy = \int_0^2 \left(\int_0^{x/2} (x-y)dy \right) dx =$$

$$\int_0^2 (xy - \frac{y^2}{2})\Big|_{y=0}^{y=x/2} dx = \int_0^2 (\frac{x^2}{2} - \frac{x^2}{8})dx = \frac{x^3}{8}\Big|_0^2 = 1.$$

Example 4. *Calculate the volume of the solid sphere S of radius r. The equation of the hollow sphere is $x^2+y^2+z^2=r^2$, but the solid whose volume we want is defined by the inequality $x^2+y^2+z^2 \leq r^2$. For fixed values $\bar{x}$ and $\bar{y}$, one has*

$$-\left(r^2-\bar{x}^2-\bar{y}^2\right)^{\frac{1}{2}} \leq z \leq \left(r^2-\bar{x}^2-\bar{y}^2\right)^{\frac{1}{2}}$$

and

$$-(r^2-\bar{x}^2)^{\frac{1}{2}} \leq y \leq (r^2-\bar{x}^2)^{\frac{1}{2}}.$$

Fubini's theorem yields

$$\textit{Volume} =$$

$$\iiint_S 1dxdydz = \int_{-r}^{r} \left(\int_{-(r^2-x^2)^{\frac{1}{2}}}^{(r^2-x^2)^{\frac{1}{2}}} \left(\int_{-(r^2-x^2-y^2)^{\frac{1}{2}}}^{(r^2-x^2-y^2)^{\frac{1}{2}}} dz \right) dy \right) dx =$$

$$\int_{-r}^{r} \left(\int_{-(r^2-x^2)^{\frac{1}{2}}}^{(r^2-x^2)^{\frac{1}{2}}} \left(2(r^2-x^2-y^2)^{\frac{1}{2}} \right) dy \right) dx =$$

$$\int_{-r}^{r} 2(r^2-x^2) \left(\int_{-1}^{1} (1-u^2)^{\frac{1}{2}} du \right) dx = \pi(r^2x - \frac{x^3}{3})\Big|_{-r}^{r} = \frac{4}{3}\pi r^3.$$

I have used here the definite integral

$$\int_{-1}^{1} (1-u^2)^{\frac{1}{2}} du = \pi/2$$

which one gets by substituting $u = \sin(t)$ in the integral (and then thinking a bit...).

Exercise 5. i. *Calculate the integral of $f(x,y) = x+y$ over the triangle with vertices $(0,0),(1,0),(1,1)$.*

ii. *Same problem, only change triangle to have vertices $(0,0),(1,0),(2,1)$. (Draw the picture and indicate how you are integrating.)*

iii. *Compute the volume of the solid tetrahedron defined by the inequalities*

$$0 \leq x \leq 1;\ 0 \leq y \leq 1;\ 0 \leq z \leq 1;\ 0 \leq x + y + z \leq 1$$

(Draw the picture.)