Math 205
Integration and calculus of several variables
week 3 - April 13, 2009

4. Coordinate Transformations

In calculating integrals in 1 dimension a very powerful tool is coordinate transformation. Given an integral $\int_a^b f(x)dx$, we choose cleverly a function $\phi(u)$. Suppose $\phi(\alpha) = a$ and $\phi(\beta) = b$. Coordinate transformation says (assuming $\phi'(u)$ doesn't change sign on the interval $[\alpha, \beta]$)

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(u))\phi'(u)du.$$

Here ϕ is viewed as a mapping $[\alpha, \beta] \to [a, b]$, $f(\phi(u))$ is the composition $f \circ \phi$ which is a function on $[\alpha, \beta]$, and $\phi'(u)$ is the "magnification factor" of the mapping ϕ. We want now to develop the same tool for integrals in several variables.

Let's start off talking about areas of parallelograms. We are thus in $\mathbb{R}^2$ and the function to be integrated is $f(x, y) = 1$. Consider the parallelogram P "spanned" by two vectors (a, c) and (b, d). That is

$$P = \{h(a, c) + k(b, d) \in \mathbb{R}^2 |\ 0 \le h, k \le 1\}.$$

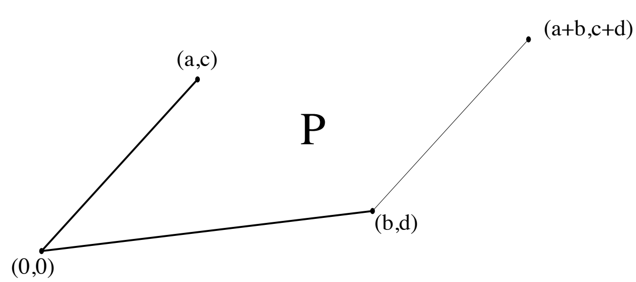

fig. 6

What is the area of the parallelogram? Let's try to figure it out by "pure thought". Given vectors v, w in the plane, let $A(v, w)$ denote the area of the corresponding parallelogram. What properties (axioms!) should we impose on A? The following seem reasonable:

- A0. $A((1, 0), (0, 1)) = 1$. (Area of unit square is 1.)
- A1. $A(v, v) = 0$. (Area of "degenerate" parallelogram with two coincident sides is zero.)
- A2. $A(cv, w) = A(v, cw) = cA(v, w)$ for $c \in \mathbb{R}$ a scalar.

A3. $A(v, w + w') = A(v, w) + A(v, w')$. similarly, $A(v + v', w) = A(v, w) + A(v', w)$.

Axiom A2 is the natural extension of the idea that e.g. doubling one of the dimensions of a rectangle doubles the area. Axiom A3 is justified by the following picture:

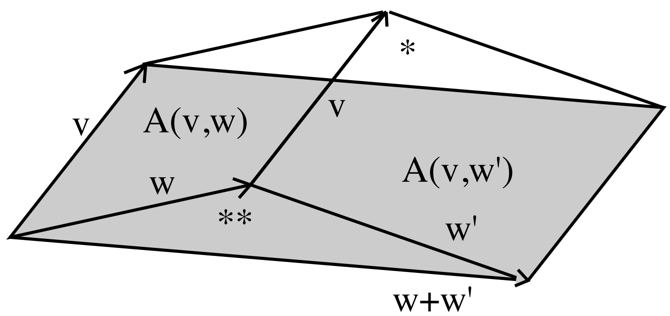

fig 7

The two narrow triangles $*$ and $**$ have the same area, so the shaded area is equal to $A(v, w) + A(v, w')$. On the other hand this shaded area is $A(v, w + w')$.

Here is a curious but important formal consequence of our axioms:

Lemma 1. *$A(v, w) = -A(w, v)$.*

Proof. We have

$$0 \overset{A1}{=} A(v + w, v + w) \overset{A3}{=} A(v, v) + A(v, w) + A(w, v) + A(w, w) = 0 + A(v, w) + A(w, v) + 0$$

which implies $A(v, w) = -A(w, v)$. □

It is computationally convenient to change notation and think of vectors in $\mathbb{R}^2$ as columns, $\binom{x}{y}$, of real numbers. We can use the axioms to compute areas as follows. Suppose we want to compute the area of the parallelogram P spanned by $\binom{a}{c}$ and $\binom{b}{d}$.

$$(4.1)\quad A(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}) = A(\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix})$$

$$= A(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}) + A(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix}) + A(\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}) + A(\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix})$$

$$= abA(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + adA(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) + bcA(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + cdA(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

$$= ad - bc.$$

What does this have to do with coordinate transformations? Consider the linear transformation

$$L : \mathbb{R}^2 \to \mathbb{R}^2; \qquad L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix};\ L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}.$$

$$L\begin{pmatrix}t\\u\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}t\\u\end{pmatrix} = \begin{pmatrix}at+bu\\ct+du\end{pmatrix}.$$

Our parallelogram P is the image of the unit square

$$U = \{(t,u) |\ 0 \le t, u \le 1\}$$

under L.

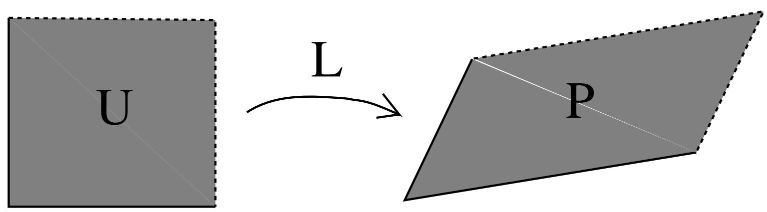

fig 8.

Since the unit square has area 1, we may say that the linear transformation $L = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ multiplies areas by $ad - bc$.

Definition 2. *The determinant,* $det(L) = det\begin{pmatrix}a & b\\c & d\end{pmatrix} = ad - bc.$

Exercise 3. *Redo the above discussion for volumes in* $\mathbb{R}^3$*. I.e. consider a function* $V(v_1, v_2, v_3)$ *for* $v_i \in \mathbb{R}^3$*. Write down axioms for* V *as above, and use these axioms to define the determinant of a matrix. The answer you should get is*

$$\det\begin{pmatrix}v_{11} & v_{12} & v_{13}\\v_{21} & v_{22} & v_{23}\\v_{31} & v_{32} & v_{33}\end{pmatrix} = v_{11}v_{22}v_{33} + v_{12}v_{23}v_{31} + v_{13}v_{21}v_{32}$$
$$- v_{13}v_{22}v_{31} - v_{12}v_{21}v_{33} - v_{11}v_{23}v_{32}.$$

In fact, the same discussion suffices to define a determinant for linear transformations in $\mathbb{R}^n$*.*

Returning to the situation in $\mathbb{R}^2$, we might be tempted to write

$$\text{Area}(P) = \iint_P 1\,dxdy \stackrel{?}{=} \iint_U \det(L)\,dtdu = \det(L). \tag{4.2}$$

However, there is a problem! Namely, the determinant can be negative.

Exercise 4. *We now have two ways to define the area of the parallelogram P spanned by vectors v and w in $\mathbb{R}^2$. One way is to take the area to be $A(v,w)$ as above. This satisfies the axioms $A0-3$ above, which is nice, but has the inconvenience that $A(v,w) = -A(w,v)$. In particular, the area depends on an "orientation". That is, it depends on ordering the vectors v, w. If we choose the wrong orientation, the area comes out negative! The other way is to approximate the area using small rectangles as in exercise 3 of section 2. The number Area(P) computed in this way is always positive. Show for P spanned by v and w that*

$$Area(P) = |A(v,w)|.$$

(Hint: Notice if you cut P up into N^2 small parallelograms just as we cut rectangles into smaller rectangles before, the area of each is constant/N^2. You find, however, that only about $4N$ of the parallelograms meet the boundary. Now try to approximate this picture using small rectangles. Use the fact that the area of a parallelogram is the height times the length of the base.)

Thus, the correct formula is

$$\text{Area}(P) = \iint_P 1dxdy \overset{!}{=} \iint_U |\det(L)|dtdu = |\det(L)|. \tag{4.3}$$

The next step is to try to derive a similar formula where the linear transformation L is replaced by a more general map $\phi : U \to R^2$. Here $U = \{(t,u)|\ 0 \le t, u \le 1\}$.

Theorem 5. *Assume ϕ is defined and $1-1$ on an open set $V \supset U$ in $\mathbb{R}^2$ and that all partial derivatives of ϕ are defined and continuous. Then the area of $\phi(U)$ is defined, and we have*

$$area(\phi(U)) = \iint_{\phi(U)} 1dxdy = \iint_U |det(d\phi_{(t,u)})|dtdu \tag{4.4}$$

Proof. Assume V is a neighborhood of $(0,0)$. Consider a small cube R with sides of length $r << 1$ in the (t,u)-plane, with lower left hand vertex at $(0,0)$. We want to compare the figures $\phi(R)$ and $d\phi_{(0,0)}(R)$. Write $\phi(t,u) = (\phi_1(t,u), \phi_2(t,u))$. Taylor's theorem says

$$\begin{aligned}\phi_i(t,u) = \partial\phi_i/\partial t(0,0)\cdot t + \partial\phi_i/\partial u(0,0)\cdot u +& \\ \phi_i^{11}(t,u)t^2 + \phi_i^{12}(t,u)tu + \phi_i^{22}(t,u)u^2&\end{aligned} \tag{4.5}$$

where the $\phi_i^{kj}(t,u)$ are continuous. In particular, since we work in a bounded region in the (u,v) plane we have

$$|\phi_i^{kj}(t,u)| < C \tag{4.6}$$

for some fixed constant C. Notice that the linear part of (4.5) can be expressed in terms of the linear transformation $d\phi_{(0,0)}$ as follows

$$d\phi_{(0,0)}\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} \partial\phi_1/\partial t(0,0) & \partial\phi_1/\partial u(0,0) \\ \partial\phi_2/\partial t(0,0) & \partial\phi_2/\partial u(0,0) \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} \partial\phi_1/\partial t(0,0)\cdot t + \partial\phi_1/\partial u(0,0)\cdot u \\ \partial\phi_2/\partial t(0,0)\cdot t + \partial\phi_2/\partial u(0,0)\cdot u \end{pmatrix}$$

We deduce from this that in the cube $0 \le t, u \le 1$

$$\mathrm{dist}(\phi(t,u), d\phi_{(0,0)}\begin{pmatrix} t \\ u \end{pmatrix}) < 3Cr^2$$

This means that if we extend the base of the parallelogram $P := d\phi_{(0,0)}(R)$ in each direction by $3Cr^2$, and if we do the same to the height, the resulting enlarged parallelogram P^+ contains $\phi(R)$. By the same token, if we decrease the base and the height, the smaller parallelogram P^- is contained in $\phi(R)$. That is

$$P^- \subset \phi(R) \subset P^+$$
$$P^- \subset P = d\phi_{(0,0)}(R) \subset P^+.$$

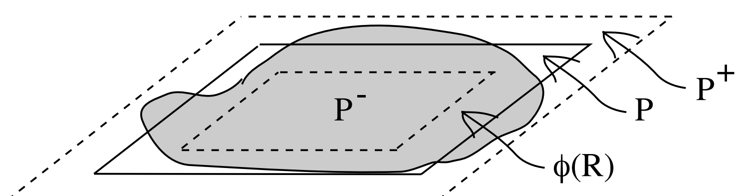

fig.9

The sides of the parallelogram P have lengths which are linear in r (why?), so

$$\mathrm{area}(P^+) - \mathrm{area}(P^-) < C'r^3$$

for some constant C' depending only on ϕ.

Now given $N \ge 1$ cut the unit square U in the (t,u)-plane into N^2 small cubes R_{pq} with side $r = 1/N$. We play the above game with each R_{pq} replacing $(0,0)$ with the lower left vertex $v_{pq} := ((p-1)/N, (q-$

1)/N) of the cube. Since there are N^2 cubes, we conclude

$$|\text{area}(\phi(R)) - \sum_{p,q} \text{area}(d\phi_{v_{pq}})(R_{pq})| =$$
$$|\sum_{p,q} \Big(\text{area}(\phi(R_{pq}) - \text{area}(d\phi_{v_{pq}})(R_{pq})\Big)| \leq C'' N^{-3} \cdot N^2 = C'' N^{-1}$$

for some constant C'' independent of N. You showed in exercise 4 that

$$\text{area}(d\phi_{v_{pq}})(R_{pq}) = |\det(d\phi_{v_{pq}})|r^2.$$

(Actually you showed this for $r = 1$, but multiplying the sides of the square by r multiplies the area of $d\phi_{v_{pq}}(R_{pq})$ by r^2.) We conclude

$$\text{(4.7)} \quad \text{area}(\phi(U)) = \lim_{N\to\infty} \sum_{p,q=1}^{N} \text{area}(\phi(R_{pq}))$$
$$= \lim_{N\to\infty} \sum_{p,q=1}^{N} |\det(d\phi_{v_{pq}})|r^2 = \iint_U |\det(d\phi_{(t,u)})| dt du$$

This completes the proof of theorem 5. □

An elaboration on these arguments yields the following basic theorem. We will not take the time to go through the details of the proof. (Recall the discussion of determinants in $\mathbb{R}^3$ and in $\mathbb{R}^n$ in exercise 3. We will only apply this theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$.)

Theorem 6. *Let $S \subset \mathbb{R}^n$ be a region, and assume that the volume of S is defined. Let $V \supset S$ be an open neighborhood of S in $\mathbb{R}^n$ and let $\phi : V \to \mathbb{R}^n$ be a mapping. Assume all partial derivatives of ϕ are defined and continuous, and that ϕ is $1-1$. Then the volume of $\phi(S)$ is defined. Let, moreover, $f : \phi(S) \to \mathbb{R}$ be a continuous function. Then*

$$\int \cdots \int_{\phi(S)} f(x_1, \ldots, x_n) dx_1 \cdots dx_n =$$
$$\int \cdots \int_S f(\phi(t_1, \ldots, t_n)) |det(d\phi_{(t_1,\ldots,t_n)})| dt_1 \cdots dt_n.$$

Example 7. *Let T be the rectangle $0 \leq x \leq \pi$; $0 \leq y \leq 2\pi$. To calculate*

$$\iint_T \cos(7x + 5y) dx dy$$

we make the substitution $\phi_1(t, u) = x = \pi t, \phi_2(t, u) = y = 2\pi u$. The matrix $D\phi(t, u)$ is the constant matrix $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 2\pi \end{smallmatrix}\right)$. We have $T = \phi(U)$

where $U: 0 \le t \le 1;\ 0 \le u \le 1$. *Thus*

$$\iint_T \cos(7x+5y)dxdy = \iint_U \cos(7\pi t + 10\pi u)|\det\begin{pmatrix}\pi & 0\\ 0 & 2\pi\end{pmatrix}|dtdu =$$

$$2\pi^2\int_0^1\left(\int_0^1 \cos(7\pi t+10\pi u)du\right)dt = 2\pi^2\int_0^1\left(\frac{\sin(7\pi t+10\pi u)}{10\pi}\Big|_{u=0}^{u=1}\right)dt$$

$$= \frac{\pi}{5}\int_0^1(\sin(7\pi t+10\pi)-\sin(7\pi t))dt = 0$$

Example 8. *Let* $T: 0 \le y \le 1;\ 0 \le x/y \le 1$. *Calculate* $\iint_T \exp(x/y)dxdy$. *Define* $\phi(t,u) = (tu,u)$. *We have* $T = \phi(U)$ *where* U *is the unit square. Also*

$$\det(d\phi_{(t,u)}) = \det\begin{pmatrix}u & t\\ 0 & 1\end{pmatrix} = u.$$

Thus

$$\iint_T \exp(x/y)dxdy = \iint_U \exp(t)udtdu = \int_0^1\left(\int_0^1 \exp(t)dt\right)udu =$$

$$\int_0^1 \exp(t)dt\int_0^1 udu = (e-1)/2.$$