Math 205
Integration and calculus of several variables

week 4 - April 20, 2009

5. Path Integration

Heretofore we have only discussed integration over domains which are in some sense "of maximal dimension". Thus, we have integrated functions over intervals in $\mathbb{R}$, over rectangles in $\mathbb{R}^2$, over cubes in $\mathbb{R}^3$ etc. Suppose we have a set $S \subset \mathbb{R}^n$ which has "dimension less than n", for example perhaps approximating the volume of S by small n-cubes gives 0 as the limit. Can we still integrate over S? The simplest example of such an S is:

Definition 1. *Let $U \subset \mathbb{R}^n$ be an open set. A path in U is a map*

$$\phi : [a, b] \to U; \qquad \phi(t) = (\phi_1(t), \ldots, \phi_n(t)).$$

We assume ϕ is differentiable in the sense that the derivatives $d\phi_i/dt$ exist and are continuous.

Example 2. *Let $\phi : [0, 2\pi] \to \mathbb{R}^2$; $\phi(t) = (\cos(t), \sin(t))$. Then ϕ is a path.*

What sort of animal can we integrate over a path? Suppose first we are in $\mathbb{R}^1 = \mathbb{R}$. We have $\phi : [a, b] \to \mathbb{R}$. Let x be the coordinate on $\mathbb{R}$ and let $f(x)$ be a continuous function. We may define

$$\int_\phi f(x)dx := \int_a^b f(\phi(t))\phi'(t)dt. \tag{5.1}$$

What about path integrals in $\mathbb{R}^n$? We do the same thing! Consider an expression of the form

$$\omega := f_1(x_1, \ldots, x_n)dx_1 + f_2(x_1, \ldots, x_n)dx_2 + \ldots + f_n(x_1, \ldots, x_n)dx_n \tag{5.2}$$

Such a thing is called a *differential* 1*-form.* Given a path $\phi : [a, b] \to \mathbb{R}^n$ we define the *path integral*

$$\int_\phi \omega := \sum_{i=1}^n \int_a^b f_i(\phi_1(t), \ldots, \phi_n(t))\phi_i'(t)dt. \tag{5.3}$$

Notice, I haven't really defined a 1-form. One may say vaguely that a 1-form is "that which can be integrated against a path to give a number".

Example 3. *Consider the* 1*-form* $\cos(y)dx + e^{xy}dy$ *on* $\mathbb{R}^2$*. Its value on the path* $\phi : [0,1] \to \mathbb{R}^2;\ x = t,\ y = t^2$ *is given by*

$$\int_\phi \cos(y)dx + e^{xy}dy = \int_0^1 (\cos(t^2) + 2e^{t^3}t)dt.$$

The crucial property of path integrals is their independence of parametrization.

Proposition 4. *Let*

$$\omega := f_1(x_1,\dots,x_n)dx_1 + f_2(x_1,\dots,x_n)dx_2 + \dots + f_n(x_1,\dots,x_n)dx_n$$

be a differential 1*-form on an open set* U *in* $\mathbb{R}^n$*. Let* $\phi : [a,b] \to U$ *be a path. Let* $\psi : [\alpha,\beta] \to [a,b]$ *with* ψ *differentiable,* $\psi(\alpha) = a,\ \psi(\beta) = b$*. Let* $\theta = \phi \circ \psi : [\alpha,\beta] \to U$*. Then*

$$\int_\phi \omega = \int_\theta \omega.$$

Proof. Let u be the coordinate on $[\alpha,\beta]$. We have

$$\begin{aligned} \int_\theta \omega &\overset{\text{def.}}{:=} \int_\alpha^\beta \sum_{i=1}^n f_i(\theta_1(u),\dots,\theta_n(u))\theta_i'(u)du \\ &\overset{\text{why?}}{=} \int_\alpha^\beta \sum_{i=1}^n f_i(\phi_1(\psi(u)),\dots,\phi_n(\psi(u)))\phi_i'(\psi(u))\psi'(u)du \qquad (5.4) \\ &\overset{\text{why?}}{=} \int_a^b \sum_{i=1}^n f_i(\phi_1(t),\dots,\phi_n(t))\phi_i'(t)dt =: \int_\phi \omega \end{aligned}$$

This proves the proposition. ∎

Exercise 5. *Compute the following path integrals* $\int_\phi \omega$*:*

i. $\phi : [0,1] \to \mathbb{R}^2;\ \phi(t) = (-t,t),\ \omega = dx_2$.
ii. $\phi : [0,2\pi] \to \mathbb{R}^2;\ \phi(t) = (\cos(t),\sin(t)),\ \omega = x_1dx_2 - x_2dx_1$.
iii. $\phi : [a,b] \to \mathbb{R}^n;\ \phi(t) = (t,t^2,\dots,t^n),\ \omega = \sum_{i=1}^n x_i dx_{n+1-i}$.

The point of view that a 1-form is that which can be integrated over a path may seem a bit vague. But actually it is quite suggestive. For example, what relations should we impose on 1-forms. Suppose to start again we are on an open set $U \subset \mathbb{R}^1$. Let x and y be two different coordinate functions on U. Suppose we have 1-forms $\omega = f(x)dx$ and $\tau = g(y)dy$ on U. When should we say $\omega = \tau$? Well, our philosophy suggests $\omega = \tau$ if and only if for *every* path $\phi : [a,b] \to U$, we have $\int_\phi \omega = \int_\phi \tau$. Now, the statement that x and y are both coordinates on

U means that e.g. $y = \psi(x)$ with $\psi'(x)$ nonvanishing on U. If the path ϕ is given by $x = \phi(t)$, then $y = \psi(\phi(t))$. We have

$$\int_\phi \tau := \int_a^b g(\psi(\phi(t)))\frac{d}{dt}(\psi(\phi(t)))dt = \int_a^b g(\psi(\phi(t)))\psi'(\phi(t))\phi'(t)dt$$

$$\int_\phi \omega = \int_a^b f(\phi(t))\phi'(t)dt$$

The two path integrals are thus equal if

$$g(\psi(\phi(t)))\psi'(\phi(t)) = g(\psi(\phi(t)))\psi'(\phi(t)).$$

Substituting $x = \phi(t)$ and $y = \psi(x)$ we find

$$f(x)dx = g(y)dy \text{ if } f(x) = g(y(x))\frac{dy}{dx} \tag{5.5}$$

Put another way, we should impose the relation on 1-forms

$$g(y(x))\frac{dy}{dx}dx = g(y)dy. \tag{5.6}$$

What about 1-forms in $\mathbb{R}^2$? Suppose we have two sets of coordinates, say x_1, x_2 and y_1, y_2. We have $y_i = y_i(x_1, x_2)$, and staring at (5.6) suggests the relations

$$dy_1 = \frac{\partial y_1}{\partial x_1}dx_1 + \frac{\partial y_1}{\partial x_2}dx_2;\ dy_2 = \frac{\partial y_2}{\partial x_1}dx_1 + \frac{\partial y_2}{\partial x_2}dx_2. \tag{5.7}$$

Thus

$$g_1(y_1, y_2)dy_1 + g_2(y_1, y_2)dy_2 = \tag{5.8}$$

$$\Big(g_1(y_1, y_2)\frac{\partial y_1}{\partial x_1}+g_2(y_1, y_2)\frac{\partial y_2}{\partial x_1}\Big)dx_1+\Big(g_1(y_1, y_2)\frac{\partial y_1}{\partial x_2}+g_2(y_1, y_2)\frac{\partial y_2}{\partial x_2}\Big)dx_2$$

Exercise 6. i. *Rewrite the differential* 1*-form* $x_1dx_1 + x_2dx_2$ *in terms of coordinates* t, u *if* $x_1 = t + u$, $x_2 = t - u$.

ii. *Suppose given* 1*-forms* $f_1dx_1 + f_2dx_2$ *and* $g_1dy_1 + g_2dy_2$ *where the* f*'s and* g*'s are related as in* (5.8)*. Show the integrals of these* 1*-forms along any path agree.*

iii. *What is the analog of* (5.8) *for* 1*-forms on* $\mathbb{R}^n$*?*