Math 205
Integration and calculus of several variables
week 4 - April 20, 2009

5. MORE ABOUT PATH INTEGRATION AND A PRACTICE MIDTERM

This is an addendum to the week 4 pdf file entitled "Path Integration". I add one new result and end with a practice midterm exam.

A *differential* 1-*form* ω on $\mathbb{R}^n$ (or on an open set $U \subset \mathbb{R}^n$) is an expression of the form $\sum_{i=1}^n f_i(x_1, \dots, x_n)dx_i$ where the f_i are "nice" $\mathbb{R}$-valued functions on $\mathbb{R}^n$ or on U.

If $\phi(t) = (\phi_1(t), \dots, \phi_n(t)) : [a, b] \to \mathbb{R}^n$ (respectively $\phi(t) = (\phi_1(t), \dots, \phi_n(t)) : [a, b] \to U$) is a path, then we can define

$$\int_\phi \omega := \sum_{i=1}^n \int_a^b f_i(\phi(t)) \frac{d\phi_i(t)}{dt} dt. \tag{1}$$

The point of this definition is that it does not depend on the parametrization of the path. If $\theta : [\alpha, \beta] \to [a, b]$ then

$$\int_{\phi\circ\theta} \omega = \int_\phi \omega.$$

Here is a way to write down lots of interesting 1-forms. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function, and define

$$\omega = dg := \frac{\partial g}{\partial x_1} dx_1 + \dots + \frac{\partial g}{\partial x_n} dx_n. \tag{2}$$

For example, if $g(x, y) = y$ then $dg = dy$. If $g(x, y) = xy$ then $dg = ydx + xdy$.

We can view d as a mapping, the *exterior derivative*

$$d : \{\text{functions}\} \to \{1 - \text{forms}\}.$$

A 1-form dg is said to be *exact*. What happens when we compute the Path integral of an exact 1-form? Good question! Glad you asked!! Remember the chain rule tells us

$$\frac{d}{dt} g(\phi(t)) = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\phi(t)) \frac{d\phi_i}{dt}.$$

Combining (1) and (2), we find the left hand identity in (3) below:

$$\int_\phi dg = \int_a^b \frac{d}{dt} g(\phi(t)) dt = g(\phi(b)) - g(\phi(a)). \tag{3}$$

The right hand identity is the fundamental theorem of calculus! Notice that the expression on the right depends only on the endpoints of ϕ. We have proven a theorem.

Theorem 1. *Let* $\phi : [a, b] \to \mathbb{R}^n$ *ba a path, and let* ω *be an exact* 1-*form on* $\mathbb{R}^n$. *Then* $\int_\phi \omega$ *depends only on the endpoints* $\phi(a), \phi(b)$. *I.e. if* $\sigma : [\alpha, \beta] \to \mathbb{R}^n$ *is another path with* $\sigma(\alpha) = \phi(a)$ *and* $\sigma(\beta) = \phi(b)$, *then* $\int_\phi \omega = \int_\sigma \omega$. *More precisely, if* $\omega = dg$, *then*

$$\int_\phi \omega = g(\phi(b)) - g(\phi(a)).$$

Corollary 2. *Let* $\phi : [a, b] \to \mathbb{R}^n$ *be a closed path; that is a path such that* $\phi(a) = \phi(b)$. *Then we have* $\int_\phi \omega = 0$ *for any exact* 1-*form* ω.

Example 3. *consider two* 1-*forms in* $\mathbb{R}^2$.

$$\omega = xdy + ydx; \quad \xi = xdy - ydx.$$

Consider the two paths from $(1, 0)$ *to* $(0, 1)$*:*

$$\phi(t) = (\cos t, \sin t),\ 0 \le t \le \pi/2; \quad \sigma(t) = (1 - t, t),\ 0 \le t \le 1.$$

Here are the path integral computations:

$$\int_\phi \omega = \int_{t=0}^{\pi/2} (\cos^2 t - \sin^2 t)dt = 0; \quad \int_\sigma \omega = \int_0^1 (1 - t)dt - tdt = 0.$$

$$\int_\phi \xi = \int_{t=0}^{\pi/2} (\cos^2 t + \sin^2 t)dt = \pi/2; \quad \int_\sigma \xi = \int_0^1 (1 - t)dt + tdt = 1.$$

Note that $\omega = xdy + ydx = d(xy)$ *while* ξ *is not an exact* 1-*form.*

Our theorem has an important converse. Let ω be a 1-form on $\mathbb{R}^n$. Suppose it is the case that the path integral $\int_\phi \omega$ depends only on the endpoints of ϕ. Said another way, suppose that whenever ϕ, σ have the same endpoints we get $\int_\phi \omega = \int_\sigma \omega$. Fix a point $p \in \mathbb{R}^n$, and define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \int_p^x \omega,$$

where the integral is computed along any path ϕ from p to x. (By assumption, it doesn't depend on the choice of path.) Write $\omega = \sum_i f_i(x_1, \dots, x_n)dx_i$. I claim $\omega = dg$. Looking back at (2), this amounts to showing $f_i = \partial g/\partial x_i$. Let

$$\psi_\varepsilon(t) = (x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n); \quad 0 \le t \le \varepsilon$$

Writing $\psi_\varepsilon(t) = (\psi_1(t), \ldots, \psi_n(t))$ we see that ψ_j is constant for $j \neq i$ and $\psi_i(t) = x_i + t$. From the definition (1) it follows that

$$\frac{\partial g}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\psi_\varepsilon} \omega = \frac{1}{\varepsilon} \int_0^\varepsilon f_i(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n) dt = f_i(x).$$

Thus, $\omega = dg$. We have proven:

Theorem 4. *Let ω be a 1-form on $\mathbb{R}^n$ and assume the path integral $\int_y^x \omega$ depends only on the endpoints x and y, and not on the path between them. Then ω is an exact 1-form. Indeed, $\omega = dg$ where $g(x) := \int_y^x \omega$ for a fixed y.*

Math 205
Practice Midterm
50 Minutes

1. Let $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be some nice function. Discuss the computation of

$$\iint_D f dx dy$$

(a) using upper and lower Riemann sums over small rectangles covering D.
(b) using Fubini's theorem.
State carefully all results that you use.

2. The map $\phi(r, \theta) = (r \cos 2\pi\theta, r \sin 2\pi\theta)$ maps the unit square to the disk D as above.
(a) Compute the derivative matrix $D\phi(r, \theta)$ and the determinant $\det D\phi(r, \theta)$.
(b) Use 2(a) to give another formula for $\iint_D f dx dy$.
(c) Use 2(b) to compute the area of D.

3. Compute the path integrals

$$\int_\phi dx + dy; \quad \int_\sigma y dx - x dy$$

Here $\phi(t) = (\cos 2\pi t, \sin 2\pi t)$ and $\sigma(t) = (t + 1, 7 - 2t)$, both on the interval $[0, 1]$.

4. Define 1-forms on $\mathbb{R}^n$. Define what it means for a 1-form to be exact. Show that the 1-form

$$\frac{xdx + ydy}{(x^2 + y^2)^2}$$

is an exact 1-form on $\mathbb{R}^2 - \{(0,0)\}$.