Math 205
Integration and calculus of several variables

week 5 - April 27, 2009

6. Chains and integration

Let I^2 be the unit square $0 \leq x, y \leq 1$. Let $\phi : I^2 \to R^2$ be a map. Suppose ϕ is $1-1$, and that $\det(D\phi)$ doesn't vanish. Let $f : R^2 \to \mathbb{R}$ be a continuous function. Our earlier work suggests one might define integration over ϕ by the formula

$$\iint_{\phi(I^2)} f(x,y)dxdy := \iint_{I^2} f(\phi_1(t,u), \phi_2(t,u))|\det(D\phi_{(t,u)})|dtdu$$

Unfortunately, we need something a bit better. We want to get rid of the absolute value signs around the jacobian and drop the assumptions about ϕ.

Let's consider an example. Suppose we had such a theory of integration. Let $f = 1$, the constant function 1, and let $\phi(t,u) = (u,t)$. We have

$$\det(D\phi_{(t,u)}) = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

Thus a theory of the sort we want leads to

$$\iint_{I^2} dudt = -\iint_{I^2} dtdu.$$

The area of the unit square is thus +1 or -1 depending on how we look at it! Ok, so to distinguish from the older theory we introduce the notation (replacing $dtdu$)

$$dt \wedge du.$$

More generally, replacing $f(t,u)dtdu$ will be *differential 2-forms*

$$f(t,u)dt \wedge du.$$

These differential forms will be the integrands in our integrals. We need also to be precise about the spaces over which integration will be performed. When we talk about spaces like I^2 or $\mathbb{R}^n$ we can specify an *orientation*. An orientation is determined by an ordering of the coordinate functions. Thus I^2, t, u and I^2, u, t are orientations on I^2. Frequently, an orientation will be understood, e.g. we usually assume I^n comes with coordinate functions $x_1, x_2, \ldots, x_n$ and the expected orientation is determined by the evident ordering of coordinate functions. Ok, now one situation in which we can calculate an integral is if we are given

i. I^2 with a chosen orientation t_1, t_2.

ii. $\phi : I^2 \to U \overset{\text{open}}{\subset} \mathbb{R}^2$ a differentiable mapping,

$$\phi(t_1, t_2) = (\phi_1(t_1, t_2), \phi_2(t_1, t_2)).$$

iii. A differential 2-form $f(x_1, x_2)dx_1 \wedge dx_2$ on U.

This integral is defined by two formulas. First, if the orientation on I^2 is given by t_1, t_2 then

$$\iint_{I^2} g(t_1, t_2) dt_1 \wedge dt_2 := \iint_{I^2} g(t_1, t_2) dt_1 dt_2 \tag{6.1}$$

In other words, the integral in this case is as we have already defined it. Second, we have the change of variable formula

$$\iint_{\phi} f(x_1, x_2) dx_1 \wedge dx_2 := \iint_{I^2} f(\phi(t_1, t_2)) \det \begin{pmatrix} \partial\phi_1/\partial t_1 & \partial\phi_1/\partial t_2 \\ \partial\phi_2/\partial t_1 & \partial\phi_2/\partial t_2 \end{pmatrix} dt_1 \wedge dt_2 \tag{6.2}$$

Example 1. *Suppose $f = 1$ and $\phi(t_1, t_2) = (t_2, t_1)$. Then* (6.2) *gives*

$$\iint_{\phi} dx_1 \wedge dx_2 = \iint_{I^2} \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dt_1 \wedge dt_2 = -\iint_{I^2} dt_1 \wedge dt_2 = -1.$$

The left hand equality uses (6.2) *and the right hand comes from* (6.1).

In our discussion of path integrals we saw that the path integral was independent of the parametrization of the path. Example 1 above shows that in higher dimension we must be careful. If our parametrization reverses the orientation, the sign of the integral can change. (Of course, in retrospect, this is true for path integrals also, but there it is clear. Reversing the orientation just means going the other way on the path, so $\int_a^b$ becomes $\int_b^a$. The notion of orientation in dimension > 1 is more subtle.)

Definition 2. *Let I^2 and J^2 be two rectangles in $\mathbb{R}^2$. We orient them by fixing coordinates t_1, t_2 on I^2 and x_1, x_2 on J^2. Let $\psi : I^2 \to J^2$ be differentiable, $x_i = \psi_i(t_1, t_2)$. We say that ψ is orientation preserving if the jacobian determinant is everywhere positive.*

$$\begin{aligned} \det(D\psi) &= \det \begin{pmatrix} \partial\psi_1/\partial t_1 & \partial\psi_1/\partial t_2 \\ \partial\psi_2/\partial t_1 & \partial\psi_2/t_2 \end{pmatrix} \\ &= \partial\psi_1/\partial t_1 \partial\psi_2/t_2 - \partial\psi_1/\partial t_2 \partial\psi_2/\partial t_1 > 0. \end{aligned}$$

ψ is an orientation preserving reparametrization if it is orientation preserving, 1-1 and onto.

Of course, the same definition works for solid rectangles in $\mathbb{R}^3$ using 3×3 determinants.

Example 3. *Suppose I^2 is oriented by choosing coordinates r, θ with $I^2 = \{(r,\theta)|\ 0 \le r \le 1;\ 0 \le \theta \le 2\pi\}$. Let $\psi(r,\theta) = (r\cos(\theta), r\sin(\theta))$. We have*

$$\det(D\psi) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r(\cos^2(\theta) + \sin^2(\theta)) = r \ge 0.$$

Note that the jacobian determinant vanishes where $r = 0$. Typically this does not cause problems in evaluating integrals because the locus $r = 0$ has zero area and can be ignored in the integral.

It will be convient to have a name for the domains over which we integrate. I shall refer to them as *chains*. The student should be aware that in more advanced work, the term chain is frequently applied to something slightly more elaborate, a sort of formal sum of what we call chains.

Definition 4. *Let $U \subset \mathbb{R}^n$ be an open set. A p-chain on U is an oriented p-rectangle I^p (typically I^p is given by $a_i \le t_i \le b_i;\ 1 \le i \le n$ for chosen coordinates t_i) and a differentiable map $\phi : I^p \to U$.*

Proposition 5. *Let ϕ be a 2-chain. Then the integral*

$$\iint_\phi f(x_1, x_2) dx_1 \wedge dx_2$$

is independent of orientation-preserving reparametrization. In other words, if $\psi : I^2 \to I^2$ is $1-1$ and onto with $\det(D\psi) > 0$ and $\theta = \phi \circ \psi$ then

$$\iint_\phi f(x_1, x_2) dx_1 \wedge dx_2 = \iint_\theta f(x_1, x_2) dx_1 \wedge dx_2$$

Proof.

$$\iint_\theta f(x_1, x_2) dx_1 \wedge dx_2 =$$

$$\iint_{I^2} f(\phi_1(\psi(t_1, t_2)), \phi_2(\psi(t_1, t_2)) \det(d(\phi \circ \psi)) dt_1 \wedge dt_2 =$$

$$\iint_{I^2} f(\phi_1(\psi(t_1, t_2)), \phi_2(\psi(t_1, t_2)) \det(D\phi(\psi(t_1, t_2)) \det(D\psi(t_1, t_2)) dt_1 \wedge dt_2 =$$

$$\iint_{I^2} f(\phi_1(u_1, u_2), \phi_2(u_1, u_2)) \det(D\phi(u_1, u_2)) du_1 \wedge du_2 =$$

$$\iint_\phi f(x_1, x_2) dx_1 \wedge dx_2.$$

We have used the chain rule for the jacobian determinant

$$\det(D(\phi \circ \psi)(t_1, t_2)) = \det(D\phi(\psi(t_1, t_2))) \det(D\psi(t_1, t_2)).$$

This completes the proof. □

The content of this proposition is that in computing our integrals, we don't have to worry about the specific parametrization as long as we keep track of the orientation.

Example 6. *Let $D = \{(x, y)|\ x^2 + y^2 \le c^2\}$ be the disk of radius c. Suppose we want to compute*

$$\iint_D x^2y^2 dx \wedge dy.$$

I haven't even bothered to give a parametrization of D. I think of the (x, y)-plane as being oriented. The content of the proposition is that any parametrization $\phi : I^2 \to D$ such that $\det(D\phi) > 0$ will do. As we have computed before, the parametrization $\phi(r, \theta) = (r\cos(\theta), r\sin(\theta))$ yields a jacobian determinant $\det(D\phi) = r \ge 0$. Again, the locus where this vanishes, the line $r = 0$, has area 0 and can be ignored, so defining $I^2 = \{(r, \theta)|\ 0 \le r \le c;\ 0 \le \theta \le 2\pi\}$ we may write

$$\iint_D x^2y^2 dx \wedge dy = \iint_{I^2} r^4 \cos^2(\theta)\sin^2(\theta) r dr d\theta =$$

$$\int_0^c r^5 dr \int_0^{2\pi} \cos^2(\theta)\sin^2(\theta) d\theta = \frac{c^6}{6}\int_0^{2\pi} \frac{\sin^2(2\theta)}{4} d\theta = \frac{c^6}{6}\frac{2\pi}{8} = \frac{c^6\pi}{24}.$$

Notice if we had changed the orientation on I^2 by interchanging r and θ, it would have made $\det(D\phi) \le 0$ and we would have gotten the wrong sign. Similarly, if we kept the orientation x, y on $\mathbb{R}^2$ but then computed $\iint_D x^2y^2 dy \wedge dx$, it would have changed the sign. On the other hand, with the switched orientation y, x, the integral $\iint_D x^2y^2 dy \wedge dx$ comes out with a plus sign! Go figure!

Exercise 7. *We have focused on integrals of 2-chains in $\mathbb{R}^2$, but integration of 3-chains in $\mathbb{R}^3$ or indeed of n-chains in $\mathbb{R}^n$ works the same way. For example, suppose we want to compute $\iiint_B dx \wedge dy \wedge dz$, where $B = \{(x, y, z)|\ x^2 + y^2 + z^2 \le c\}$. We fix the natural orientation on $\mathbb{R}^3$, and consider spherical coordinates*

$$\phi(r, \sigma, \theta) = (r\cos(\sigma), r\sin(\sigma)\cos(\theta), r\sin(\sigma)\sin(\theta)).$$

Show that if we define

$$I^3 = \{(r, \sigma, \theta)|\ 0 \le r \le c;\ 0 \le \sigma \le \pi;\ 0 \le \theta \le 2\pi\}$$

then the determinant of the jacobian is $2r^2\sin(\sigma)$*, which is nonnegative on* I^3*. Use this parametrization to compute the integral.*

Finally, in this section, I want to consider integration of 2-chains in $\mathbb{R}^3$. The "yoga" is the same. A 2-chain in $\mathbb{R}^3$ is a mapping $\phi : I^2 \to \mathbb{R}^3$ of an oriented rectangle I^2 into $\mathbb{R}^3$. A 2-form is an expression

$$\omega = f_{12}(x_1,x_2,x_3)dx_1 \wedge dx_2 + f_{23}(x_1,x_2,x_3)dx_2 \wedge dx_3 + f_{13}(x_1,x_2,x_3)dx_1 \wedge dx_3. \tag{6.3}$$

By definition

$$\iint_\phi \omega = \sum_{1\le i<j\le 3} \iint_\phi f_{ij}(x_1,x_2,x_3)dx_i \wedge dx_j \tag{6.4}$$
$$= \sum_{1\le i<j\le 3} \iint_{I^2} f_{ij}(\phi_1(t_1,t_2),\phi_2(t_1,t_2),\phi_3(t_1,t_2)) \det \begin{pmatrix} \frac{\partial\phi_i}{\partial t_1} & \frac{\partial\phi_i}{\partial t_2} \\ \frac{\partial\phi_j}{\partial t_1} & \frac{\partial\phi_j}{\partial t_2} \end{pmatrix} dt_1 dt_2$$

Notice there is no "$\wedge$" between the last dt_1dt_2. At this point we have used our orientation and the ordering $dx_i \wedge dx_j$ to determine the order of the rows and columns of the jacobian determinant (and hence to fix its sign). Our integral now is just the usual one which can be computed with Fubini.

Formula (6.4) is complicated but important. To keep things straight, you can calculate as follows

$$\begin{aligned} dx_i \wedge dx_j|_{x_i=\phi_i,x_j=\phi_j} &= d\phi_i \wedge d\phi_j = \\ &(\partial\phi_i/\partial t_1 dt_1 + \partial\phi_i/\partial t_2 dt_2) \wedge (\partial\phi_j/\partial t_1 dt_1 + \partial\phi_j/\partial t_2 dt_2) = \\ &\partial\phi_i/\partial t_1 \partial\phi_j/\partial t_1 dt_1 \wedge dt_1 + \partial\phi_i/\partial t_1 \partial\phi_j/\partial t_2 dt_1 \wedge dt_2 + \\ &\partial\phi_i/\partial t_2 \partial\phi_j/\partial t_1 dt_2 \wedge dt_1 + \partial\phi_i/\partial t_2 \partial\phi_j/\partial t_2 dt_2 \wedge dt_2. \end{aligned} \tag{6.5}$$

Now we are forced to set $dt_k \wedge dt_k = 0$ (why?) and $dt_1 \wedge dt_2 = -dt_2 \wedge dt_1$, so the computation becomes

$$dx_i \wedge dx_j|_{x_i=\phi_i,x_j=\phi_j} = \det \begin{pmatrix} \frac{\partial\phi_i}{\partial t_1} & \frac{\partial\phi_i}{\partial t_2} \\ \frac{\partial\phi_j}{\partial t_1} & \frac{\partial\phi_j}{\partial t_2} \end{pmatrix} dt_1 \wedge dt_2. \tag{6.6}$$

Bearing in mind that we have oriented I^2 so $dt_1 \wedge dt_2$ becomes just dt_1dt_2, we see that (6.6) is exactly the contribution of $dx_i \wedge dx_j$ in (6.4).

Example 8. *Let* $I^2 = \{(r,\theta)|\ 0 \le r \le 1;\ 0 \le \theta \le 2\pi\}$*. Define*

$$\phi : I^2 \to \mathbb{R}^3; \qquad \phi(r,\theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1-r^2}).$$

We would like to compute the integral $\iint_\phi dx_2 \wedge dx_3$. *We have*

$$\iint_\phi dx_2 \wedge dx_3 = \iint_{I^2} \det \begin{pmatrix} \frac{\partial(r\sin(\theta))}{\partial r} & \frac{\partial(r\sin(\theta))}{\partial \theta} \\ \frac{\partial(\sqrt{1-r^2})}{\partial r} & \frac{\partial(\sqrt{1-r^2})}{\partial \theta} \end{pmatrix} dr \wedge d\theta$$

$$= \iint_{I^2} \det \begin{pmatrix} \sin(\theta) & r\cos(\theta) \\ \frac{-r}{\sqrt{1-r^2}} & 0 \end{pmatrix} dr \wedge d\theta = \int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}} \int_0^{2\pi} \cos(\theta) d\theta = 0.$$

Exercise 9. *Show using the same sort of argument as in proposition 5 that* $\iint_\phi \omega$ *in* (6.4) *is independent of orientation-preserving reparametrization of* ϕ.

Exercise 10. *Let* $I^2 = \{(t_1, t_2) \mid 0 \leq t_1, t_2 \leq 1\}$. *Orient* I^2 *so* $\int_{I^2} dt_1 \wedge dt_2 > 0$. *Let* $\phi : I^2 \to \mathbb{R}^3$ *be given by*

$$\phi(t_1, t_2) = (t_1 + t_2, t_1 - t_2, t_1 t_2).$$

Let $\omega = dx_1 \wedge dx_2 + dx_2 \wedge dx_3$. *Compute* $\int_\phi \omega$.

Exercise 11. *Let* I^2 *be as in the previous exercise. Let* $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{R}^3$ *be two points. Define* $\phi : I^2 \to \mathbb{R}^3$ *by* $\phi(t_1, t_2) = t_1 a + t_2 b$. *Compute* $\int_\phi dx_1 \wedge dx_2$.

Exercise 12. *Let* $I^2 = \{(t_1, t_2) \mid 0 \leq t_1 \leq \pi;\ 0 \leq t_2 \leq 2\pi\}$. *Orient* I^2 *so* $\int_{I^2} dt_1 \wedge dt_2 > 0$. *Define* $\phi : I^2 \to \mathbb{R}^3$ *by* $\phi(t_1, t_2) = (\cos t_1 \sin t_2, \sin t_1 \sin t_2, \cos t_2)$. *Draw a picture of the surface parametrized by* ϕ. *Compute* $\int_\phi dx_1 \wedge dx_2 + dx_2 \wedge dx_3$.

Exercise 13. *Let* $I^2 = \{(t_1, t_2) \mid -1 \leq t_1 \leq 1;\ -1 \leq t_2 \leq 1\}$ *be a square, and let* $\phi : I^2 \to \mathbb{R}^3$ *be given by*

$$\phi(t_1, t_2) = (\phi_1(t_1, t_2), \phi_2(t_1, t_2), \phi_3(t_1, t_2)),$$

where the ϕ_i *are all differentiable. Suppose* $\phi(0, 0) = (0, 0, 0)$. *Consider the Jacobian matrix*

$$D\phi(0,0) = \begin{pmatrix} \frac{\partial \phi_1}{\partial t_1}(0,0) & \frac{\partial \phi_1}{\partial t_2}(0,0) \\ \frac{\partial \phi_2}{\partial t_1}(0,0) & \frac{\partial \phi_2}{\partial t_2}(0,0) \\ \frac{\partial \phi_3}{\partial t_1}(0,0) & \frac{\partial \phi_3}{\partial t_2}(0,0) \end{pmatrix}.$$

If the two column vectors in this matrix are linearly independent, then give an intuitive explanation why they span the tangent plane at $(0, 0, 0)$ *to the surface parametrized by* ϕ. *(Hint: how intuitively do you pass from the vector* $(\phi_1(t_1, t_2), \phi_2(t_1, t_2), \phi_3(t_1, t_2))$ *to the vector* $(\frac{\partial \phi_1}{\partial t_1}(0,0), \frac{\partial \phi_2}{\partial t_1}(0,0) \frac{\partial \phi_3}{\partial t_1}(0,0))$*?)*