Math 205
Integration and calculus of several variables

week 6 - May 4, 2009

8. Fiddle d

Have you ever asked yourself, "What is dx?" In the middle of doing your math homework, do you ever feel like the Coyote in Roadrunner cartoons who's just chased the bird off a cliff and looked down to realize there is no solid earth beneath his feet? What does an expression like $f(x)dx$ *mean* anyway?

Let $U \subset \mathbb{R}^n$ be an open set. Let $f : U \to \mathbb{R}$ be a differentiable function. Consider the following problem: find a 1-form (which we will call) $d(f)$ on U such that for any path $\phi : [a, b] \to U$, the path integral is given by

$$\int_\phi d(f) = f(\phi(b)) - f(\phi(a)). \tag{8.1}$$

Example 1. *Suppose $n = 1$. I claim we may take $d(f) := f'(x)dx$. Indeed, with this definition, the path integral just becomes*

$$\begin{aligned}\int_\phi d(f) &= \int_\phi f'(x)dx \\ &= \int_a^b f'(\phi(t))\phi'(t)dt = \int_a^b \frac{d}{dt} f(\phi(t))dt = f(\phi(b)) - f(\phi(a)).\end{aligned}$$

The case for general $\mathbb{R}^n$ is not much more difficult. We simply define

$$d(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i. \tag{8.2}$$

Proposition 2. *Let $\phi : [a, b] \to \mathbb{R}^n$ be a path. Then*

$$\int_\phi d(f) = \int_\phi \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = f(\phi(b)) - f(\phi(a)).$$

Proof. Write $\phi(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Then

$$\int_\phi \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i =$$

$$\int_a^b \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{dx_i(t)}{dt} dt = \int_a^b \frac{d}{dt}(f(\phi(t))dt = f(\phi(b)) - f(\phi(a)).$$

□

Example 3. *In* $\mathbb{R}^2$*, let* $f(x,y) = \sqrt{x^2+y^2}$*. Then*

$$d(f) = \frac{xdx + ydy}{\sqrt{x^2+y^2}}.$$

We have for any path ϕ*,*

$$\int_\phi \frac{xdx + ydy}{\sqrt{x^2+y^2}} = \sqrt{\phi_1(b)^2 + \phi_2(b)^2} - \sqrt{\phi_1(a)^2 + \phi_2(a)^2}.$$

It is important to note that the path integral condition (8.1) uniquely determines $d(f)$. Indeed, if $D(f)$ were another solution satisfying (8.1) for every path ϕ we could write $d(f) - D(f) = \sum g_i(x_1, \ldots, x_n)dx_i$ and conclude that

$$0 = \sum_i \int_\phi g_i(\phi(t))\phi_i'(t)dt.$$

If we take for ϕ the path $\phi(t) = (c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n)$ (t in the i-th coordinate, c_j constants) we see that

$$\int_a^b g_i(c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n)dt = 0$$

Since the constants c_j and the endpoints a and b are arbitrary, we conclude that all the $g_i(x_1, \ldots, x_n) = 0$, so $D(f) = d(f)$.

The following definition is convenient.

Definition 4. *A* 0*-form on* $U \subset \mathbb{R}^n$ *is a differentiable function* $f : U \to \mathbb{R}$*.*

Thus, we have defined a map

$$d : \{0 - \text{forms}\} \to \{1 - \text{forms}\} \tag{8.3}$$

$$d(f) := \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

Finally, the wiley reader will have detected disaster ahead. After all, we have a 1-form dx_i already. But x_i is a function on $U \subset \mathbb{R}^n$, so we may define in our new sense a 1-form $d(x_i)$. Fortunately, our coyote is on solid ground, because

$$d(x_i) = \sum_j \frac{\partial x_i}{\partial x_j} dx_j = dx_i.$$

For the philosophically inclined in the peanut gallery, this gives a rigorous meaning to the "infinitesimal" dx.