

Math 205

Integration and calculus of several variables

week 6 - May 4, 2009

9. MORE d FIDDLE

There is a general pattern here. We will define maps for all p

$$(9.1) \quad d : \{p - \text{forms}\} \rightarrow \{(p + 1) - \text{forms}\}.$$

Before doing that, it is best to think carefully on the level of chains in order to understand precisely the property our map d should satisfy. We had defined a p -chain to be a differentiable map $\phi : I^p \rightarrow \mathbb{R}^n$, where I^p was a p -rectangle in \mathbb{R}^p . To simplify notation a bit, I will assume henceforth that I^p is the unit cube

$$I^p = \{(t_1, \dots, t_p) \mid 0 \leq t_i \leq 1\}.$$

On the other hand, in the spirit of the great French philosopher Descartes, whose maxim: “pourquoi faire simple quand on peut faire compliqué?” has inspired mathematicians ever since, we will generalize the notion of chain, to permit formal linear combinations of differentiable maps. In other words, given differentiable maps $\phi_i : I^p \rightarrow \mathbb{R}^n$ and $c_i \in \mathbb{R}$, we will talk about the p -chain

$$\phi := \sum_i c_i \phi_i$$

What does this mean? It does *not* mean to add the maps ϕ_i in any vector sense. Remember that p -chains and p -forms have this sort of symbiotic relation. A form is that which can be integrated over a chain, and a chain is that over which a form can be integrated. Ok, so by definition

$$(9.2) \quad \int_{\phi} \omega := \sum_i c_i \int_{\phi_i} \omega.$$

Now we want to set up a “chain game” which is in some sense *dual* to our form game. Recall we have defined 0-forms to be functions. Let us define

Definition 1. A 0-chain p in \mathbb{R}^n is a finite, formal linear combination of points in \mathbb{R}^n

$$p := \sum_i c_i [p_i]; \quad p_i \in \mathbb{R}^n.$$

Remember this is a *formal* linear combination. Don't actually try to add the points. To "integrate" a 0-form over a 0-chain, we simply define

$$\int_p f := \sum_i c_i f(p_i).$$

(It occurs to me that the word *formal* in mathematics has a meaning similar to its meaning in fashion. You may write down a formal sum but you never actually *add* one. In the same way you may own a formal dress but you never actually *wear* it.)

Now we define a mapping

$$(9.3) \quad \partial : \{1\text{-chains}\} \rightarrow \{0\text{-chains}\}$$

$$\partial(\phi) := \phi(1) - \phi(0); \quad \partial\left(\sum_i c_i \phi_i\right) := \sum_i c_i (\phi_i(1) - \phi_i(0)).$$

Our result about $d(f)$ can be rewritten

$$(9.4) \quad \int_{\phi} df = \int_{\partial\phi} f.$$

A mathematician would say the mappings d and ∂ are *dual* to each other.

Exercise 2. 1. Compute df for the following functions

- i. $x^2 + y^2$
- ii. $\cos(x) \sin(y)$
- iii. $e^x + e^y + e^z$

2. For each df from problem 1, compute $\int_{\phi} df$ for the paths $\phi(t) = (t, t)$; $0 \leq t \leq 1$ and $\phi(t) = (\cos(t), \sin(t))$; $0 \leq t \leq 2\pi$.

We now consider 2-chains. To begin with, we define a 1-chain ∂I^2 on I^2 as follows.

$$(9.5) \quad \partial I^2 = (t, 0) + (1, t) - (t, 1) - (0, t)$$

Here for example $(t, 0)$ is short for the path $\phi(t) = (t, 0)$; $0 \leq t \leq 1$. The picture is:

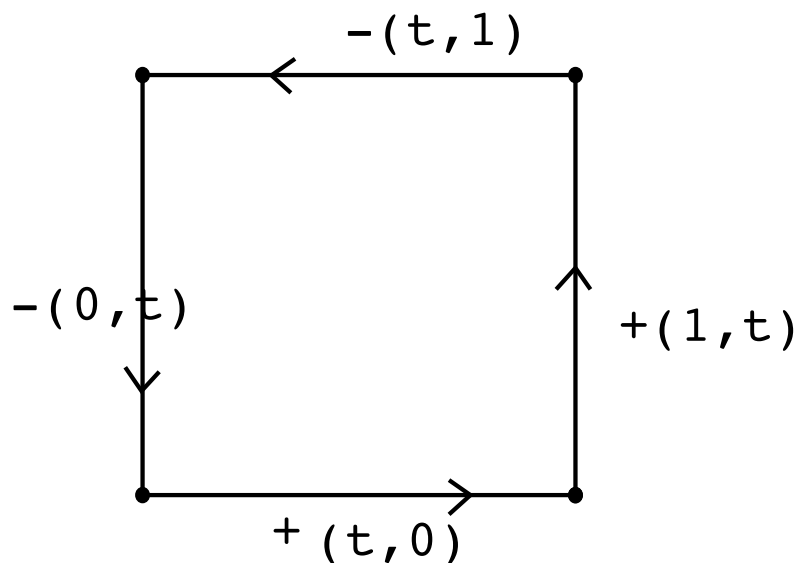


fig. 10

Example 3. Let $\omega = 5t_1dt_2 + 7t_2dt_1$. We compute

$$\begin{aligned} \int_{\partial I^2} \omega &= \int_{(t,0)} \omega + \int_{(1,t)} \omega - \int_{(t,1)} \omega - \int_{(0,t)} \omega \\ &= 7 \int_0^1 0dt + 5 \int_0^1 1dt - 7 \int_0^1 1dt - 5 \int_0^1 0dt = -2 \int_0^1 dt = -2. \end{aligned}$$

Definition 4. Given a differentiable map $\phi : I^2 \rightarrow \mathbb{R}^n$, we can define a 1-chain

$$\partial\phi := \phi(t, 0) + \phi(1, t) - \phi(t, 1) - \phi(0, t).$$

More generally, if $\phi = \sum_i c_i \phi_i$ is a 2-chain as above, where $\phi_i : I^2 \rightarrow \mathbb{R}^n$, we define

$$\partial\phi = \sum_i c_i \partial\phi_i.$$

Here, for example, $\phi(t, 0)$ denotes the path $t \mapsto \phi(t, 0)$; $0 \leq t \leq 1$.

The picture is

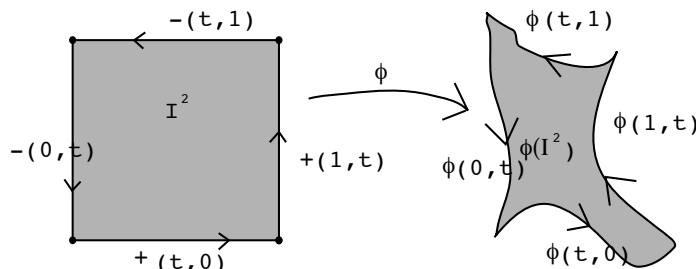


fig. 11

Example 5. Suppose $\phi : I^2 \rightarrow \mathbb{R}^3$ is the 2-chain

$$\phi(t_1, t_2) = (t_2^2, t_1 t_2, t_1^2)$$

Then $\partial\phi$ is the 1-chain

$$\phi(t, 0) + \phi(1, t) - \phi(t, 1) - \phi(0, t) = (0, 0, t^2) + (t^2, t, 1) - (1, t, t^2) - (t^2, 0, 0).$$

Here again, note we do not add these as vectors or anything of the sort.

What the notation means is that given a 1-form ω , we have

$$\int_{\partial\phi} \omega = \int_{(0,0,t^2)} \omega + \int_{(t^2,t,1)} \omega - \int_{(1,t,t^2)} \omega - \int_{(t^2,0,0)} \omega.$$

If, for example, $\omega = t_1 t_2 dt_3$, we get

$$\int_{\partial\phi} \omega = - \int_0^1 t d(t^2) = - \int_0^1 2t^2 dt = -\frac{2}{3}.$$

Exercise 6. 1. Compute $\partial\phi$ for the following $\phi : I^2 \rightarrow \mathbb{R}^3$:

$$\phi(t_1, t_2) = (t_1 + t_2, t_1 - t_2, 1); \quad \phi(t_1, t_2) = (e^{t_1}, e^{t_2}, \cos(t_1 t_2))$$

2. Show the composition

$$\{2\text{-chains}\} \xrightarrow{\partial} \{1\text{-chains}\} \xrightarrow{\partial} \{0\text{-chains}\}$$

is zero.

Next step is to define the map d from 1-forms to 2-forms as in (9.1). We define

$$(9.6) \quad d\left(f_1(t_1, t_2)dt_1 + f_2(t_1, t_2)dt_2\right) := \left(\frac{\partial f_2}{\partial t_1} - \frac{\partial f_1}{\partial t_2}\right)dt_1 \wedge dt_2.$$

Exercise 7. 1. Compute $d\omega$ for the following 1-forms ω .

$$t_1 dt_2 + t_2 dt_1; \quad e^x dy; \quad dr + r \cos(\theta) d\theta.$$

2. Let f be a differentiable function on an open set containing I^2 .
Show

$$d(d(f)) = 0.$$

Proposition 8. Let ω be a 1-form on an open set in \mathbb{R}^2 containing I^2 .
Then

$$\iint_{I^2} d\omega = \int_{\partial I^2} \omega.$$

Proof. We just compute

$$\begin{aligned} (9.7) \quad \iint_{I^2} d\omega &= \iint_{I^2} \left(\frac{\partial f_2}{\partial t_1} - \frac{\partial f_1}{\partial t_2} \right) dt_1 \wedge dt_2 \\ &= \int_0^1 dt_2 \int_0^1 \frac{\partial f_2}{\partial t_1} dt_1 - \int_0^1 dt_1 \int_0^1 \frac{\partial f_1}{\partial t_2} dt_2 \\ &= \int_0^1 (f_2(1, t_2) - f_2(0, t_2)) dt_2 - \int_0^1 (f_1(t_1, 1) - f_1(t_1, 0)) dt_1. \end{aligned}$$

Also

$$\begin{aligned} (9.8) \quad \int_{\partial I^2} \omega &= \int_{\partial I^2} f_1 dt_1 + f_2 dt_2 \\ &= \int_{(t,0)} f_1 dt_1 + \int_{(1,t)} f_2 dt_2 - \int_{(t,1)} f_1 dt_2 - \int_{(0,t)} f_2 dt_2 \\ &= \int_0^1 f_1(t_1, 0) dt_1 + \int_0^1 f_2(1, t_2) dt_2 - \int_0^1 f_1(t_1, 1) dt_1 - \int_0^1 f_2(0, t_2) dt_2. \end{aligned}$$

The proposition follows by comparing (9.7) and (9.8). \square

Finally, Let's prove the result for an arbitrary 2-chain in \mathbb{R}^2 .

Theorem 9 (Green's Theorem). Let ω be a 1-form on an open set $U \subset \mathbb{R}^2$. Let ϕ be a 2-chain on U . Then

$$\iint_{\phi} d\omega = \int_{\partial\phi} \omega.$$

Proof. We may write $\phi = \sum_i c_i \phi_i$ where $\phi_i : I^2 \rightarrow U$. By definition 4 we have $\partial\phi = \sum_i c_i \partial\phi_i$. Thus

$$\begin{aligned} \iint_{\phi} d\omega &= \sum_i c_i \iint_{\phi_i} d\omega \\ \int_{\partial\phi} \omega &= \sum_i c_i \int_{\partial\phi_i} \omega. \end{aligned}$$

so it suffices to prove the theorem in the case $\phi : I^2 \rightarrow U$. Write

$$\begin{aligned}\phi(t_1, t_2) &= (x_1(t_1, t_2), x_2(t_1, t_2)) \\ \omega &= f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2 = \omega_1 + \omega_2; \quad \omega_i := f_i dx_i \\ d\omega &= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2 = d\omega_1 + d\omega_2 \\ d\omega_1 &= -\frac{\partial f_1}{\partial x_2} dx_1 \wedge dx_2; \quad d\omega_2 = \frac{\partial f_2}{\partial x_1} dx_1 \wedge dx_2.\end{aligned}$$

To simplify the calculation, let's again break things up and notice

$$\begin{aligned}\iint_{\phi} d\omega &= \iint_{\phi} d\omega_1 + \iint_{\phi} d\omega_2 \\ \int_{\partial\phi} \omega &= \int_{\partial\phi} \omega_1 + \int_{\partial\phi} \omega_2\end{aligned}$$

Thus it will be enough to prove the theorem with $\omega = \omega_i$. (Having proved the theorem for ω_1 and ω_2 , we simply add the results to conclude the theorem for ω .) We will prove the theorem for ω_1 , and leave as an exercise for the reader to prove the theorem for ω_2 .

So, suppose $\omega = f_1 dx_1$. We compute

$$\begin{aligned}(9.9) \quad \iint_{\phi} d\omega &= \iint_{\phi} -\frac{\partial f_1}{\partial x_2} dx_1 \wedge dx_2 \\ &= \iint_{I^2} -\frac{\partial f_1}{\partial x_2} \det \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \end{pmatrix} dt_1 \wedge dt_2 \\ &= -\iint_{I^2} \frac{\partial f_1}{\partial x_2} \left(\frac{\partial x_1}{\partial t_1} \frac{\partial x_2}{\partial t_2} - \frac{\partial x_1}{\partial t_2} \frac{\partial x_2}{\partial t_1} \right) dt_1 dt_2\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\partial\phi} \omega &= \int_{\partial\phi} f_1 dx_1 = \int_{\phi(t,0)} f_1 dx_1 + \int_{\phi(1,t)} f_1 dx_1 - \int_{\phi(t,1)} f_1 dx_1 - \int_{\phi(0,t)} f_1 dx_1 \\ &= \int_0^1 f_1(x(t_1, 0)) \frac{\partial x_1}{\partial t_1} dt_1 + \int_0^1 f_1(x(1, t_2)) \frac{\partial x_1}{\partial t_2} dt_2 \\ &\quad - \int_0^1 f_1(x(t_1, 1)) \frac{\partial x_1}{\partial t_1} dt_1 - \int_0^1 f_1(x(0, t_2)) \frac{\partial x_1}{\partial t_2} dt_2 \\ &\stackrel{(*)}{=} \int_{\partial I^2} f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_1} dt_1 + f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_2} dt_2.\end{aligned}$$

(Note: the identity (*) is crucial. The reader should check it carefully.)
 We may now apply proposition 8 to this last path integral

$$\begin{aligned}
 (9.10) \quad & \int_{\partial I^2} f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_1} dt_1 + f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_2} dt_2 = \\
 & \iint_{I^2} d\left(f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_1} dt_1 + f_1(x(t_1, t_2)) \frac{\partial x_1}{\partial t_2} dt_2\right) = \\
 & \iint_{I^2} \left(-\frac{\partial f_1(x(t))}{\partial t_2} \frac{\partial x_1}{\partial t_1} - f_1(x(t)) \frac{\partial^2 x_1}{\partial t_1 \partial t_2} + \frac{\partial f_1(x(t))}{\partial t_1} \frac{\partial x_1}{\partial t_2} + \right. \\
 & \quad \left. f_1(x(t)) \frac{\partial^2 x_1}{\partial t_1 \partial t_2} \right) dt_1 dt_2 \\
 & = \iint_{I^2} \left(-\frac{\partial f_1(x(t))}{\partial t_2} \frac{\partial x_1}{\partial t_1} + \frac{\partial f_1(x(t))}{\partial t_1} \frac{\partial x_1}{\partial t_2} \right) dt_1 dt_2 \\
 & = \iint_{I^2} \left(-\left(\frac{\partial f_1(x(t))}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial f_1(x(t))}{\partial x_2} \frac{\partial x_2}{\partial t_2} \right) \frac{\partial x_1}{\partial t_1} \right. \\
 & \quad \left. + \left(\frac{\partial f_1(x(t))}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial f_1(x(t))}{\partial x_2} \frac{\partial x_2}{\partial t_1} \right) \frac{\partial x_1}{\partial t_2} \right) dt_1 dt_2 \\
 & = - \iint_{I^2} \frac{\partial f(x(t))}{\partial x_1} \left(\frac{\partial x_1}{\partial t_1} \frac{\partial x_2}{\partial t_2} - \frac{\partial x_1}{\partial t_2} \frac{\partial x_2}{\partial t_1} \right) dt_1 dt_2.
 \end{aligned}$$

Now just note that the bottom lines of (9.9) and (9.10) are the same. \square