Math 205
Midterm
50 Minutes

1(a). State Fubini's theorem and use it to calculate the area of the triangle T in $\mathbb{R}^2$ with vertices $(1,0),\ (0,0),\ (0,3)$. Draw the picture to illustrate what you are doing.
(b). Let $\phi(x,y) = (x(1-y), 3y)$. Compute $\det D\phi(x,y)$ and use it to give another computation of the area of T. State the change of coordinate result you are using to justify your result.
(c) Compute $\iint_T xydxdy$.

2(a). Define carefully the following: (i) Path in $\mathbb{R}^n$. (ii) Differential 1-form in $\mathbb{R}^n$. (iii) Differential 1-form $dg(x_1,\ldots,x_n)$ (exact differential 1-form). (iv) Path integral.
(b). What is the important invariance property of the path integral of an arbitrary 1-form? What further invariance property holds for 1-forms dg?

3(a). Compute the path integrals

$$\int_\phi \frac{-xdx + ydy}{xy}; \quad \int_\sigma ydx - xdy.$$

Here $\phi(t) = (\cos 2\pi t, \sin 2\pi t)$ and $\sigma(t) = (t+1, 7-2t)$, both on the interval $[0,1]$.
(b). (This problem is more subtle. I suggest you do it last.) The expression

$$\eta := \sqrt{dx_1^2 + \ldots + dx_n^2}$$

is not a differential 1-form. Nonetheless, given a path $\phi : [a,b] \to \mathbb{R}^n$, give a definition for

$$\int_\phi \eta.$$

Show that your definition is independent of parametrization. I.e. if $\theta : [\alpha,\beta] \to [a,b]$ with $\theta(\alpha) = a,\ \theta(\beta) = b$, and $\theta'(u) \geq 0,\ u \in [\alpha,\beta]$, then show that $\int_{\phi\circ\theta} \eta = \int_\phi \eta$. Compute $\int_\phi \eta$ for $\phi : [0,2\pi] \to \mathbb{R}^2,\ \phi(t) = (\cos t, \sin t)$. What geometric interpretation can you suggest for $\int_\phi \eta$?