Math 205
Summary, June 1, 2009

I. Before Midterm

A. Integration in $\mathbb{R}^n$

$U \subset \mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$$\int_{\phi(U)} f dy_1 dy_2 \cdots dy_n = \int_U |\det(D\phi)| f(\phi(x)) dx_1 dx_2 \cdots dx_n$$

Warning: Need ϕ to be 1 to 1 on U. This integral does not track orientation.

B. Fubini's thm.

$$\int_{I^2} f(x,y) dx dy = \int_I dx \int_I f(x,y) dy$$

and generalizations.

II. Path Integration $\phi : [a,b] \to \mathbb{R}^n$ and $\omega = \sum f_i dx_i$.

$$\int_\phi \omega = \int_a^b \sum f_i(\phi(t)) \frac{dx_i}{dt} dt$$

A. Exact forms $\omega = df$

$$\int_\phi df = f(b) - f(a).$$

B. Arc length

$$ds = \sqrt{\sum \left(\frac{dx_i}{dt}\right)^2} dt; \quad \phi(t) = (x_1(t), \ldots, x_n(t))$$

III. Higher degree forms and chains

A. p-forms $\sum f_{i_1 i_2 \ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$

Calculus of p-forms: $dx \wedge dy = -dy \wedge dx,\ dx \wedge dx = 0.$

B. Exterior derivative

$$d(f dx_1 \wedge \cdots \wedge dx_p) = (-1)^p \sum_{i=p+1}^{n} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_p \wedge dx_i.$$

C. Orientation on I^p; $\int_1^p dt_1 \wedge \cdots \wedge dt_p = +1.$

Boundary: $\partial I = (1) - (0).$

$$\partial I^p = \partial I \times I^{p-1} - I \times \partial I \times I^{p-2} + \ldots + (-1)^{p-1} I^{p-1} \times \partial I.$$

D. Stokes's thm on I^n. ω an $n-1$-form on I^n.

$$\int_{I^n} d\omega = \int_{\partial I^n} \omega.$$

E. Stokes's thm, general form

$$\int_{\phi(I^p)} d\omega = \int_{\phi(\partial I^p)} \omega.$$

F. $d \circ d = 0$.

Exact forms and closed forms. Examples of closed forms which are not exact.

IV. Geometry and Applications

A. dot and cross products of vectors: geometric interpretation

$$v \cdot w = |v||w|\cos\theta; \quad v \times w = z,\ z \cdot v = 0 = z \cdot w,\ |z| = |v||w|.$$

B. Vector fields

Normal vector field. $\phi : I^2 \to \mathbb{R}^3$

$$\vec{N} = (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}) \times (\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}); \quad \vec{n} = \vec{N}/|\vec{N}|,\ d\sigma = |\vec{N}|dtdu$$

C. Div, $Grad$, $Curl$.

$$\phi : I^3 \to \mathbb{R}^3;\ \omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

$$\int_\phi d\omega = \int_\phi Div(f_1, f_2, f_3) dx_1 \wedge dx_2 \wedge dx_3$$

$$\phi : I^2 \to \mathbb{R}^3;\ \omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

$$\int_\phi d\omega = \int_\phi Curl(f_1, f_2, f_3) \cdot \vec{n} d\sigma$$

Analogous formulas in $\mathbb{R}^2$.

D. Geometric interpretation of Div and $Curl$ for integrals.

E. Applications

fluid flow in $\mathbb{R}^2$.

Maxwell's equations