ALGEBRAIC CYCLES AND ADDITIVE CHOW GROUPS

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Abstract. A report on recent results and outstanding problems concerning additive chow groups.

"Mighty oaks from little acorns grow."

1. Introduction

Additive algebraic K-theory means roughly "replace the algebraic group GL by the lie algebra $\mathfrak{g}l$ where ever you see it", [L]. Given the central role of algebraic cycles in motivic cohomology, one may ask for an algebraic cycle interpretation of additive K-theory. More than a simple restatement of the theory, such a geometric reformulation suggests new problems. What are motivic sheaves over $k[t]/(t^2)$? What is the tangent space to the space of motives?

One should, I believe, have the following picture in mind. An algebraic circle is represented by the pair $\mathbb{A}^1, \{0, t\}$ for any $t \neq 0$. It is natural, geometrically, to think of the limiting situation $t \to 0$ as represented by

$$\mathbb{A}^1, 2(0) \tag{1.1}$$

As a simple example, for k a field

$$H^1_M(k, \mathbb{Z}(1)) = Pic(\mathbb{A}^1_k, \{0, t\}) = \mathbb{G}_m(k) = k^\times, \tag{1.2}$$

It is natural to write for the corresponding additive group

$$TH^1_M(k, \mathbb{Z}(1)) := Pic(\mathbb{A}^1_k, 2(0)) = \mathbb{G}_a(k) = k. \tag{1.3}$$

A word of warning. The notation TH_M suggests tangent space, but this is perhaps not the precise analogy. A better picture would be a sort of non-semi-stable degeneration, with a group like $\mathbb{G}_m$ which is constant in the parameter t degenerating to an additive group for $t = 0$.

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2. Hyperbolic versus Euclidean

Another analogy is the picture of the hyperbolic metric on a disk of expanding radius. As the radius tends to ∞, the metric tends to the euclidean metric. The link with motivic cohomology is summarized by the following 4-term exact sequence

$$\begin{array}{ccccccccc} 0 & \to H^1_M(\mathbb{C},\mathbb{Z}(2)) & \to & B_2(\mathbb{C}) & \xrightarrow{\delta} & \mathbb{C}^\times \otimes \mathbb{C}^\times & \to K_2(\mathbb{C}) & \to 0 \\ & \downarrow{\scriptstyle reg} & & \downarrow{\scriptstyle vol} & & & & \\ & \mathbb{R} & \xrightarrow{=} & \mathbb{R} & & & & \end{array} \tag{2.1}$$

Here $B_2(\mathbb{C})$ has generators $[x]_2,\ x \in \mathbb{C}-\{0,1\}$, and $vol[x]_2$ is the hyperbolic volume of the tetrahedron in hyperbolic 3-space with vertices at infinity at the points $0, 1, \infty, x \in \mathbb{P}^1(\mathbb{C})$. One defines $\delta[x]_2 = x \otimes (1-x)$, and the symbol $[x]_2 \in B_2$ satisfies the classical 5-term relation.

The additive analogue of 2.1 was worked out in [BE] using a K-theoretic definition of the additive motivic group $TH^1_M(k,\mathbb{Z}(2))$ involving the relative K-group $K_2(\mathcal{O}_{\mathbb{A}^1,0},(t^2))$:

(2.2)

$$\begin{array}{ccccccccc} 0 & \to TH^1_M(k,\mathbb{Z}(2)) & \to & TB_2(k) & \xrightarrow{T\delta} & k \otimes k^\times & \xrightarrow{a\otimes b \mapsto adb/b} & \Omega^1_k & \to 0 \\ & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & & & & \\ & k & \xrightarrow{=} & k & & & & & \end{array}$$

By definition, $TB_2(k)$ is a $k^\times$-module (not a k-vector space!) with generators $\langle x\rangle_2$ which satisfy the 4-term information-theory equation first identified in this context by Kontsevich

$$\langle x\rangle_2 - \langle y\rangle_2 + x \star \langle y/x\rangle_2 + (1-x) \star \langle (1-y)/(1-x)\rangle_2 = 0 \tag{2.3}$$

The regulator map ρ in (2.2) is defined by $\rho\langle x\rangle_2 = x(1-x)$. One might hope that $\rho\langle x\rangle_2$ represents some Euclidean polyhedron with Euclidean volume $x(1-x)$. The action of $k^\times$ on the target of ρ is by the cube of the standard character, suggesting an interpretation in terms of volume in $\mathbb{R}^3$.

3. Cycles

We consider algebraic cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^n$, where $\mathbb{A}^1$ has parameter t and the $\mathbb{P}^1$ have parameters t_i, $1 \le i \le n$. We have divisors $\sigma_i : t_i = 1$ and $\delta_i = (t_i)_0 - (t_i)_\infty$. Fix $m \ge 2$. A closed subvariety $Z \subset \mathbb{A}^1 \times (\mathbb{P}^1)^n$ will be said to be congruent to 1 mod t^m if scheme-theoretically

$$Z \cdot \{t^m = 0\} \subset \bigcup_{i=1}^{n} Z \cdot \sigma_i \tag{3.1}$$

We consider algebraic cycles $z = \sum n_\nu Z_\nu$ such that all faces $Z_\nu \cdot \delta_I$ (I multi-index) have codim. $\geq \#(I)$ in Z_ν and are congruent to 1 mod m. Write

$$T_m CH^p(k,q) = T_m H_M^{2p-q}(k, \mathbb{Z}(p)) \tag{3.2}$$

for the resulting higher Chow groups, generated by cycles of codim. p on $\mathbb{A}^1 \times (\mathbb{P}^1)^{q-1}$. The case of 0-cycles has been worked out by K. Rülling.

Theorem 3.1 (K. Rülling, [R])**.** $T_m H_M^n(k, \mathbb{Z}(n)) \cong W_{m-1}\Omega^{n-1}$, *the de Rham-Witt groups built from the "big" Witt ring on* k*.*

Remark 3.2. Rülling uses a slightly different version of the congruence condition (3.1). I have checked casually that his results hold under (3.1), but this should perhaps be verified more carefully.

A cycle-theoretic version of (2.2) involves cycles of dimension 1. This has been considered by J. Park [P]. His main result is the construction of a non-trivial regulator map

$$\rho_{m,n} : T_m CH^{n-1}(k,n) \to \Omega_k^{n-3}. \tag{3.3}$$

(Of particular interest is the case $n = 3$, $\rho_{m,3} : T_m CH^2(k,3) \to k$.)

4. An abstract regulator construction

In trying to generalize Park's regulator construction to cycles of dim. $r > 1$, one is led to consider meromorphic differential forms

$$\frac{t_{i_0} - 1}{t^{m+1}} dt_{i_1}/t_{i_1} \wedge \ldots \wedge dt_{i_r}/t_{i_r} \tag{4.1}$$

on $\mathbb{A}_t^1 \times (\mathbb{P}_{t_i}^1)^n$. The most subtle aspect of his work, the careful control of signs necessary to verify that the regulator he constructs is trivial on boundaries of two-dimensional cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n+1}$, will not be attempted here; but let me sketch the construction of a generalized regulator on cycles.

Let X be a smooth, projective variety, and let $D_0, \ldots, D_r$ be effective Cartier divisors on X. Let $Z \subset X$ be a closed subvariety of dimension r. We assume all intersections $Z \cdot D_I$ are either empty or have the correct dimension. In particular, $Z \cap \bigcap_0^r D_i = \emptyset$.

For $z \in Z$ a closed point, we have surjections

$$H_z^r(Z, \Omega_{Z/k}^r) \twoheadrightarrow H^r(Z, \Omega_{Z/k}^r) \overset{\deg}{\twoheadrightarrow} k \tag{4.2}$$

(The group on the left is local cohomology. We do not assume Z smooth.)

Let ω be a meromorphic Kähler r-form on X which is regular on $X - \bigcup_0^r D_i$. (For example, if $X = \mathbb{A}^1_t \times (\mathbb{P}^1_{t_i})^n$ one might take $\omega = \frac{t_{i_0}-1}{t^{m+1}} dt_{i_1}/t_{i_1} \wedge \ldots \wedge dt_{i_r}/t_{i_r}$ with $D_0 : t = 0$ and $D_j = (t_{i_j})_0 + (t_{i_j})_\infty$, $1 \leq j \leq r$.

Chose $i \neq j \in [0, r]$ and define $X(ij) = (D_i + D_j) \cap \bigcap_{h \neq i,j} D_h$ (resp. $Z(ij) = X(ij) \cap Z$). The r open sets $X - (D_i + D_j),\ X - D_h,\ h \neq i, j$ cover $X - X(ij)$, and we may view ω as a Cech $r-1$ cocycle representing a class $\omega(ij) \in H^{r-1}(X - X(ij), \Omega^r_{X/k})$ (resp. by restriction $\omega_Z(ij) \in H^{r-1}(Z - Z(ij), \Omega^r_{Z/k})$.)

$Z(ij)$ is a finite set of points which we can write as a disjoint union $Z(ij) = Z_j(i) \amalg Z_i(j)$, where $Z_j(i) \subset Z - Z \cap D_j$. Write $\deg_j(i) \in k$ for the image of $\omega_Z(ij)$ under the composition

$$H^{r-1}(Z - Z(ij), \omega^r) \xrightarrow{\partial} H^r_{Z(ij)}(Z, \omega^r) \xrightarrow{proj} H^r_{Z_j(i)}(Z, \omega^r) \xrightarrow{(4.2)} k \tag{4.3}$$

Clearly, $\deg_j(i) = -\deg_i(j)$.

Now take $\nu \in [0, r]$ with ν, i, j all distinct. Note the sets $Z_j(\nu)$ and $Z_j(i)$ coincide. (They are the intersection of Z with all the $D_h,\ h \neq j$.) Furthermore, the open coverings

$$\begin{aligned} &X - (D_i + D_j),\ X - D_h,\ h \neq i, j \\ &X - (D_\nu + D_j),\ X - D_h,\ h \neq \nu, j \end{aligned} \tag{4.4}$$

agree upto reordering as open coverings of $X - X(ij) - D_j = X - X(\nu j) - D_j$. It follows that

$$\deg_i(j) = -\deg_j(i) \overset{(*)}{=} \pm \deg_j(\nu) = \mp \deg_\nu(j). \tag{4.5}$$

But now, for a fourth index μ, the identity (*) above gives

$$\deg_i(j) = \pm \deg_\nu(j) = \pm \deg_\nu(\mu) \tag{4.6}$$

We conclude that, upto a sign which depends on i, j and the ordering of the divisors D_h, the quantity

$$\deg_j(i) \in k \tag{4.7}$$

is independent of the choice of i, j.

Example 4.1. When Z is a curve ($r = 1$) the poles of $\omega|_Z$ are supported on $Z \cap (D_0 + D_1)$. Our construction then amounts to taking residues along those poles lying on $Z \cap D_0$.

References

[BE] Bloch, S., Esnault, H., The Additive Dilogarithm, Documenta Math. Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), pp. 131-155

[BE2] Bloch, S., Esnault, H., An additive version of higher Chow groups, Annales Scientifiques de l'École Normale Supérieure, 4-o série, **36** (2003), 463-477.

[L] Loday, J.-L., Cyclic Homology. Grundlehren der Mathematischen Wissenschaften **301**. Springer-Verlag, Berlin, 1998. xx+513 pp.

[P] Park, J., Infinitesimal and Cyclic Aspects of Motives, Thesis, Univ. Chicago, (2006).

[R] Rülling, K., Additive Chow groups with higher modulus and the generalized de Rham-Witt complex, 74 pages (2005), to appear in J. Alg. Geom.

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