Periods Associated to Algebraic Cycles

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Outline

Motivic Cohomology and K-theory

- Beilinson definition

$$H^p_M(X, \mathbb{Q}(q)) := gr^q_\gamma K_{2q-p}(X)_\mathbb{Q}.$$

  - Example:

$$H^{2p}_M(X, \mathbb{Q}(p)) = gr^p_\gamma K_0(X) \cong CH^p(X)_\mathbb{Q}$$

Higher Chow Groups

- $\Delta^n_k := \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\sum t_i - 1)$ algebraic n-simplex.
- $\iota_i : \Delta^{n-1} \hookrightarrow \Delta^n$ locus $t_i = 0$.
- $\mathcal{Z}^p(X \times \Delta^n)' \subset \mathcal{Z}^p(X \times \Delta^n)$ cycles in good position w.r.t. faces.
- $\delta_i := \iota_i^* : \mathcal{Z}^p(X \times \Delta^n)' \to \mathcal{Z}^p(X \times \Delta^{n-1})'; \delta = \sum(-1)^i \delta_i$
- Complex $\mathcal{Z}^p(X, \cdot)$:

$$\cdots \xrightarrow{\delta} \mathcal{Z}^p(X \times \Delta^n)' \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{Z}^p(X \times \Delta^1)' \xrightarrow{\delta} \mathcal{Z}^p(X)$$

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Higher Chow Groups and Motivic Cohomology

- X smooth, $H^p_M(X, \mathbb{Z}(q)) \cong CH^q(X, 2q-p)$.
  - Variant: Cubical cycles: $\square := \mathbb{P}^1 - \{1\}$; Replace Δ^n with $\square^n$; factor out by degeneracies.
  - Face maps $\iota^j_i : \square^{n-1} \hookrightarrow \square^n, j = 0, \infty$

Examples

- Chow groups: $CH^p(X,0) = CH^p(X) = H_M^{2p}(X,\mathbb{Z}(p))$.
- Units: $CH^1(X,1) = H_M^1(X,\mathbb{Z}(1)) = \Gamma(X,\mathcal{O}_X^\times)$.
- Milnor classes: $f_1,\ldots,f_n \in \Gamma(X,\mathcal{O}_X^\times)$. $\{f_1,\ldots,f_n\} := [(x,f_1(x),\ldots,f_n(x)) \cap (X \times \square^n)] \in CH^n(X,n) = H_M^n(X,\mathbb{Z}(n))$.
- $\dim X = 2$, $C_i \subset X$ curves, $f_i \in k(C_i)^\times$ rational functions. $\Gamma_i := \{(c,f_i(c)) | c \in C_i\} \in \mathcal{Z}^2(X \times \square^1)$.

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Higher Chow DGA

- $X = \mathrm{Spec}\, k$ a point. Product

$$\mathcal{Z}^p(\square^n) \otimes \mathcal{Z}^q(\square^m) \to \mathcal{Z}^{p+q}(\square^{m+n}).$$

- $\mathfrak{N}^p(r) := \mathcal{Z}^r(\square_k^{2r-p})_{\mathbb{Q},Alt}$
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Cycles and the Tannakian Category of Mixed Tate Motives

- Hopf algebra $H := H^0(Bar(\mathfrak{N}^*(\bullet)))$
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Example: Dilogarithm Motive

- $$\begin{array}{ccc} \mathfrak{N}^1(1)\otimes\mathfrak{N}^1(1) & \xrightarrow{\text{mult}} & \mathfrak{N}^2(2) \\ \uparrow\partial & & \uparrow\partial \\ (\mathfrak{N}^1(1)\otimes\mathfrak{N}^1(0))\oplus(\mathfrak{N}^1(0)\otimes\mathfrak{N}^1(1)) & \longrightarrow & \mathfrak{N}^1(2)/\partial\mathfrak{N}^0(2) \end{array}$$
- $\mathfrak{N}^1(1)/\partial\mathfrak{N}^1(0)\cong k^\times\otimes\mathbb{Q}$
- $\mathfrak{N}^1(2)/\partial\mathfrak{N}^0(2)\ni T_x, x\in k-\{0,1\}$ Totaro cycles
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- $T_x = \{(t, 1-t, 1-xt^{-1}) \mid t \in \mathbb{P}^1\}$ parametrized curve in $\square^3$.
- $\partial T_x = (x, 1-x) \in \mathcal{Z}^2(\square^2) = \mathfrak{N}^2(2)$.
- $[(x) \otimes (1-x), T_x] \in H^0(Bar(\mathfrak{N}^*(\bullet)))$
- Comodule generated is $Dilog(x)$.
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Extensions associated to Cycles

- X smooth variety dim. d over $\mathbb{C}$. $Z = \sum n_i Z_i \in \mathcal{Z}^p(X)$ algebraic cycle. Write $|Z| = \bigcup_i Z_i$.
- Betti cohomology sequence

$$0 \to H^{2p-1}(X, \mathbb{Q}(p)) \to H^{2p-1}(X - |Z|, \mathbb{Q}(p)) \xrightarrow{\partial} H_{2d-2p}(|Z|, \mathbb{Q}(p-d)) \xrightarrow{cl} H^{2p}(X, \mathbb{Q}(p))$$

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- Extension of Hodge structures

$$0 \to H^{2p-1}(X, \mathbb{Q}(p)) \to \partial^{-1}(\mathbb{Q} \cdot Z) \to \mathbb{Z} \to 0$$

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- $Z \in \mathcal{Z}^p(X \times \Delta^q)$ meeting faces properly. Assume $\partial_i Z = 0 \in \mathcal{Z}^p(X \times \Delta^{q-1}), \forall i$.
- $[Z] \in H^{2p-q}(X, \mathbb{Z}(p))$. Example:

$$Z = \{(x, x)\} \subset (\mathbb{P}^1 - \{0, \infty\}) \times \square^1, \ [Z] \in H^1(\mathbb{G}_m, \mathbb{Z}(1))$$

- If X is projective and $q \geq 1$, or if $q \geq p$, then $[Z]$ is torsion.
- when $[Z]$ torsion, same construction, working with $(X \times \Delta^q - |Z|, X \times \partial\Delta^q - |Z| \cap X \times \partial\Delta^q)$ yields

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$$Z = \{(x,x)\} \subset (\mathbb{P}^1 - \{0,\infty\}) \times \square^1, \ [Z] \in H^1(\mathbb{G}_m, \mathbb{Z}(1))$$

- If X is projective and $q \geq 1$, or if $q \geq p$, then $[Z]$ is torsion.
- when $[Z]$ torsion, same construction, working with $(X \times \Delta^q - |Z|, X \times \partial\Delta^q - |Z| \cap X \times \partial\Delta^q)$ yields

$$0 \to H^{2p-1}(X \times \Delta^q, X \times \partial\Delta^q; \mathbb{Q}(p)) \to M_Z \to \mathbb{Q} \to 0$$

- Get class in $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2p-q-1}(X, \mathbb{Q}(p)))$

Extensions associated to Cycles IV

$$\mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), H) \cong H_{\mathbb{C}}/(F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

- H a pure Hodge structure; $F^* H_{\mathbb{C}}$ Hodge filtration.
- $0 \to H \to M \to \mathbb{Z}(0) \to 0$;
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Examples; Chow groups

- (Intermediate Jacobians) $Z \in \mathcal{Z}^p(X)$, $[Z] = 0 \in H^{2p}(X, \mathbb{Z}(p)) \rightsquigarrow$

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- (Biextensions) $\dim X = d,\ p + q = d + 1,\ Z \in \mathcal{Z}^p(X),\ V \in \mathcal{Z}^q(X),\ [Z] = 0 = [V],\ |Z| \cap |V| = \emptyset$
Construct $M_{Z,V}$ subquotient of $H^{2p-1}(X - |Z|, |V|; \mathbb{Q}(p))$

$$W_2 M_{Z,V} = \mathbb{Q}(1); \quad gr^W_{-1} M_{Z,V} \text{ pure weight } -1; \quad gr^W_0 M_{Z,V} = \mathbb{Q}(0).$$

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Examples: Higher Chow groups

- $X = \mathrm{Spec}\, K$ a point.
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  - $[K:\mathbb{Q}] = d = r_1 + 2r_2,\ p \geq 2$.

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- X a curve
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- Rigidity; degenerating families of cycles.
  - X_t degenerating family of elliptic curves.
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A Final Example: Mahler Measure Extension

- $P \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}],\ \mathbb{G}_m^n \supset V : P = 0.$
- $\Gamma_P = \{(z_1, \ldots, z_n; z_1, \ldots, z_n, P(z)) \in (\mathbb{G}_m^n - V) \times \square^{n+1}\}.$

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Real regulators and Amplitudes Associated to Extensions

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extension of Hodge structures.

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Family of cycles parametrized by $t \rightsquigarrow$

multi-valued function $amp(*_t)$ with variation $\in \langle \omega, H_{\mathbb{Z}} \rangle$.

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- Sunset (or sunrise) graph; 2 vertices and 3 edges.
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  m_j mass associated to j-th edge.

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- $E_t \cap \Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$ plus 3 other points.
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Sunset Amplitude

- $Li_2(x) := \sum x^n/n^2$ dilogarithm.
- $$A = 2\pi i(\text{rational multiple of periods of } E_t) + \frac{6\varpi_r(t)}{\pi} E_\Theta(q)$$
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Where did the cycle go?

- Milnor symbol $\{X/Z, Y/Z\} \in H^2_M(E_t - S, \mathbb{Z}(2))$.
- Because $S := \mathfrak{h} \cap E_t \subset E_t(tors)$, symbol extends to $H^2_M(E_t, \mathbb{Z}(2))$.
- Amplitude $\leftrightarrow$ regulator of this symbol.
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Hasse-Weil L-functions

- $X/\mathrm{Spec}\,\mathbb{Q}$ projective, smooth.
- ℓ-adic cohomology group $H^q_{et}(\overline{X}, \mathbb{Q}_\ell)$.
- Hasse-Weil L-function ($I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ = inertia subgroup at p; $F_p =$ geo. frobenius; $\ell \neq p$)

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The Real Involution(s)

- $X/\mathbb{R}$.
- 3 involutions:
  - $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.
  - $conj : H^*_{Betti}(X, \mathbb{C}) \to H^*_{Betti}(X, \mathbb{C})$
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- de Rham conjugation (H^*_{DR} defined algebraically)

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The Real Involution(s)

- $X/\mathbb{R}$.
- 3 involutions:
  - $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.
  - $conj : H^*_{Betti}(X, \mathbb{C}) \to H^*_{Betti}(X, \mathbb{C})$
  - $\overline{F}_\infty := F_\infty \circ conj = conj \circ F_\infty$.
- de Rham conjugation (H^*_{DR} defined algebraically)

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Volume Form

- $X/\operatorname{Spec}\mathbb{Q}$ smooth, projective, geometrically connected.
- $n > \frac{q}{2} + 1$, $H_{\mathbb{Z}} := H^q_{Betti}(X_{\mathbb{C}}, \mathbb{Z}(n))$ Hodge structure with $\overline{F}_\infty$ action.

$$G := \left(H_{\mathbb{C}}/(F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})\right)^{\overline{F}_\infty = +1}$$

- G is abelian Lie group with tangent space

$$T_{G,\mathbb{R}} := H_{DR}(X/\mathbb{R})(n)/F^0 H_{DR}(X/\mathbb{R})(n) = \left(H_{DR}(X/\mathbb{Q})(n)/F^0 H_{DR}(X/\mathbb{Q})(n)\right) \otimes \mathbb{R} =: T_{G,\mathbb{Q}} \otimes \mathbb{R}$$

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- $H_M^{q+1}(X, \mathbb{Z}(n))_{\mathbb{Z}} \subset H_M^{q+1}(X, \mathbb{Z}(n))$; classes with everywhere good reduction.
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$$H_M^{q+1}(X, \mathbb{Z}(n))_{\mathbb{Z}} \xrightarrow{Ext.cl.} G$$
- Conjecture(Beilinson) (i) The extension class map is injective modulo torsion with image discrete in G.
(ii) The rank of $H_M^{q+1}(X, \mathbb{Z}(n))_{\mathbb{Z}}$ equals the order of zero of $L(H^q, s)$ at $q + 1 - n$.
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- For X/F, F numberfield, the conjecture is formulated by taking G_σ for the various $\mathbb{R}$- and $\mathbb{C}$-embeddings of F.
- Beilinson conjecture is true for $X = \mathrm{Spec}\, F$ a number field. (Borel).
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- $$F_{A,B,C}(q) = \sum_{n \in \mathbb{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}n^t A n + n^t B + C}}{(q)_{n_1} \cdots (q)_{n_r}}$$
- $A \in M_r(\mathbb{Q})$ symmetric, > 0, $B \in \mathbb{Q}^r$, $C \in \mathbb{Q}$. $(q)_n := (1-q)\cdots(1-q^n)$.
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- T_{Q_i} Totaro cycle

$$\partial(\sum_{i=1}^{r} T_{Q_i}) = \prod_i (Q_i \otimes \prod_j Q_j^{A_{ij}}) = 1 \in \bigwedge^2 \mathbb{C}^\times \otimes \mathbb{Q}$$

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Regulator Computation

$$0 \to \mathbb{C}^\times_\mathbb{Q} \xrightarrow{a \mapsto 2\pi i \otimes a} \mathbb{C} \otimes \mathbb{C}^\times \xrightarrow{\exp \otimes id} \mathbb{C}^\times \otimes \mathbb{C}^\times \to 0$$

Lemma

Expression

$$\varepsilon(a) := [\log(1-a) \otimes a] + \left[2\pi i \otimes \exp\left(\frac{-1}{2\pi i}\int_0^a \log(1-t)\frac{dt}{t}\right)\right] \in \mathbb{C} \otimes \mathbb{C}^\times$$

is well-defined independent of the choice of a path from 0 *to* a*. We have* $(\exp \otimes id)\varepsilon(a) = (1-a) \otimes a$*.*

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Regulator and Nahm's Conjecture

Example

$\sum_{i=1}^{r}(\varepsilon(Q_i) - \varepsilon(1 - Q_i)) \in \mathbb{C}_{\mathbb{Q}}^{\times} \subset \mathbb{C} \otimes \mathbb{C}^{\times}$

Definition

Rogers dilogarithm $L(x) := Li_2(x) + \frac{1}{2}\log(x)\log(1-x), 0 < x < 1.$ $L(1) = \pi^2/6$. *Here* $Li_2(x) = \sum x^n/n^2$. *Note* $L(x) + L(1-x) = \pi^2/6$

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Consider the compact piece of the regulator

$$H^1_M(K, \mathbb{Q}(2)) \xrightarrow{reg} \mathbb{C}^\times_{\mathbb{Q}} = \mathbb{R} \oplus S^1_{\mathbb{Q}} \to S^1_{\mathbb{Q}}.$$

If we identify $\mathbb{R}/\pi^2\mathbb{Q} = S^1_{\mathbb{Q}}$ *by* $x \mapsto \exp(x/2\pi i)$, *then* $reg(\sum T_{Q_i}) \equiv \sum L(Q_i) \mod \mathbb{Q}\pi^2$

Proposition

Given $A \in M_r(\mathbb{Q})$ *with* A *symmetric,* > 0*, a necessary condition for there to exist* $B \in \mathbb{Q}^r, C \in \mathbb{Q}$ *such that* $F_{A,B,C}(q)$ *is modular is* $\sum L(Q_i) \in \mathbb{Q}\pi^2$.

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Regulator and Nahm's Conjecture III

Corollary

(i) If $\sum T_{Q_i} \in H^1_M(K, \mathbb{Q}(2))$ *vanishes, then for any* $B \in \mathbb{Q}^r, C \in \mathbb{Q}$*,* $F_{A,B,C}(q)$ *has the correct asymptotics as* $q \to 1$ *to be a modular function.*

(ii) The Q_i *are algebraic and real. If they are totally real, then (i) holds.*

- Jacobian matrix for system $1 - x_i = \prod_{j=1}^r x_j^{A_{ij}}$ is invertible at $x_i = Q_i$ so Q_i algebraic.
- One has examples where Q_i not totally real and $\sum T_{Q_i}$ is not torsion.

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