Hodge Classes and Deformation of Cycles

Spencer Bloch

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Albert Lectures, University of Chicago

Outline

Hodge Classes in Families

Joint work with H. Esnault and M. Kerz.

- X/S smooth projective family.
  - Char. 0, $S = \overline{\mathbb{Q}}[[t]]$ or $S = \mathbb{C}[[t]]$.
  - Mixed Char., $S = \mathrm{Spec}\, W$, $W = W(k)$ ring of Witt Vectors. k perfect, char. p.
- $S = \overline{\mathbb{Q}}[[t]]$; Gauß-Manin connection

$$\nabla : H^*_{DR}(X/S) \to H^*_{DR}(X/S) \otimes \Omega^1_{\overline{\mathbb{Q}}[[t]]}$$

$$H^*_{DR}(X/S) \cong H^*_{DR}(X/S)^{\nabla=0} \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[[t]]$$

$$H^*_{DR}(X/S)^{\nabla=0} \cong H^*_{DR}(Y/\overline{\mathbb{Q}}); \quad Y = X \times_S \mathrm{Spec}\, \overline{\mathbb{Q}}$$

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Hodge Classes in Families, II

- $z = [Z] \in H^{2r}_{DR}(Y/\overline{\mathbb{Q}})$ class of an algebraic cycle.
- z extends uniquely to horizontal class $\tilde{z} \in H^{2r}_{DR}(X/S)^{\nabla=0}$.
- In general, $\tilde{z} \notin F^r H^{2r}_{DR}(X/S)$.
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$$z = [Z] \in H^{2r}_{crys}(Y/W) \cong H^{2r}_{DR}(X/W)$$

- Again $z \notin F^r H^{2r}_{DR}(X/W)$

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Continuous Cohomology and K-theory

- X/S smooth projective formal scheme.
- $S = \mathrm{Spf}\, R$; $S_n = \mathrm{Spec}\, R_n$; $X_n = X \times_R R_n$. $X_\bullet$ = ind-system
- $R = \overline{\mathbb{Q}}[[t]]$ or $R = W(k)$; k perfect char. p; $R_n = R/\mathfrak{m}_R^n$.
- Prosystem of Nisnevich sheaves $\{\mathbb{Z}_{X_\bullet}(r)\}$ (motivic complex)
- Continuous K-theory $K_{X_\bullet}$ pro-system of simplicial presheaves (Quillen)

$$K_i^{cont}(X_\bullet) := [S^i_{X_1}, K_{X_\bullet}].$$

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The Chern Character

$$\begin{array}{ccccccccc}
0 \to & \varprojlim^1_n K_1(X_n) & \to & K_0^{cont}(X_\bullet) & \to & \varprojlim_n K_0(X_n) & \to 0 \\
& \cong \downarrow ch & & \cong \downarrow ch & & \cong \downarrow ch & \\
0 \to & (\bigoplus_r \varprojlim^1_n H^{2r-1}(X_1, \mathbb{Z}_{X_\bullet}(r)))_{\mathbb{Q}} & \to & \bigoplus_r CH^r_{cont}(X_\bullet)_{\mathbb{Q}} & \to & (\bigoplus_r \varprojlim_n H^{2r}(X_1, \mathbb{Z}_{X_\bullet}(r)))_{\mathbb{Q}} & \to 0
\end{array}$$

- Crucial point: Thomason descent for K-theory of singular schemes. $K_0(X_n)$ is the Grothendieck group of vector bundles on X_n as explained in the first lecture.

Chern Classes (Recall)

- $\mathcal{V}_n$ on X_n rank r vector bundle generated by global sections
- $s_1, \ldots, s_p$ general sections of $\mathcal{V}_n$. Concrete possibility to talk about algebraic cycle $c_{r-p+1}(\mathcal{V}_n)$.
- Lifting $\mathcal{V}_n$ to $\mathcal{V}_{n+1}$ on X_{n+1} would yield lifted chern class.
- In the limit, $\varprojlim \mathcal{V}_n$ can be algebrized.
- The bad news: We can only lift $[\mathcal{V}_n] \in K_0(X_n)$. $\varprojlim[\mathcal{V}_n]$ cannot be algebrized. Only get classes to all infinitesimal orders.

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Hodge Classes in Families, the Main Theorem

Theorem

X/S smooth projective formal scheme; $S = Spf(R)$. R complete dvr.
(i) Assume $R = \overline{\mathbb{Q}}[[t]]$, and write $X_n = X \times_R \mathrm{Spec}\, R/t^n R$. Let $z = [Z]_{DR} \in H^{2r}_{DR}(X_1/\overline{\mathbb{Q}})$ be an algebraic cycle class. Then $\tilde{z} \in H^{2r}_{DR}(X/R)^{\nabla=0}$ lies in $F^r H^{2r}_{DR}(X/R)$ if and only if $[Z] \in CH^r(X_1)_{\mathbb{Q}}$ lifts to $CH^r_{cont}(X)_{\mathbb{Q}}$.
(ii) Assume $R = W(k)$. Assume further $\dim X_1 < p - 6$. Let $z = [Z]_{crys} \in H^{2r}_{crys}(X_1/W) \cong H^{2r}_{DR}(X/W)$ be an algebraic cycle class. Then $z \in F^r H^{2r}_{DR}(X/R)_{\mathbb{Q}}$ if and only if $[Z] \in CH^r(X_1)_{\mathbb{Q}}$ lifts to $CH^r_{cont}(X)_{\mathbb{Q}}$.
(iii) Assume $R = \mathbb{C}[[t]]$. Assume further that the Kunneth projectors are algebraic for $H^*_{DR}(X_\eta \times X_\eta)$ where $\eta \to \mathrm{Spec}\, \mathbb{C}[[t]]$ is the generic point. Then $\tilde{z} \in F^r H^{2r}_{DR}(X/S)$ iff there exists a class $\mathcal{Z} \in CH^r_{cont}(X_\bullet)$ such that $\tilde{z} = [\mathcal{Z}]_{DR} \in F^r H^{2r}_{DR}(X/S)$.

Discussion

- What the theorem says in case $R = \overline{\mathbb{Q}}[[t]]$:
  A cycle class $[Z] \in CH^r(X_1)_{\mathbb{Q}}$ lifts in the sense that there exists $\zeta \in (\varprojlim K_0(X_n))_{\mathbb{Q}}$ with $ch(\zeta)|X_1 = [Z]$ if and only if the horizontal lifting of $[Z]_{DR}$ lies in $F^r H^{2r}_{DR}(X/R)$.
- What the theorem *does not say* in case $R = \overline{\mathbb{Q}}[[t]]$:
  "Hodgeness" of the horizontal lifting of $[Z]_{DR}$ implies existence of a lifting to X or to some algebrization of X.

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  Assume $\dim X_1 < p - 6$. A cycle class $[Z] \in CH^r(X_1)_{\mathbb{Q}}$ lifts in the sense that there exists $\zeta \in (\varprojlim K_0(X_n))_{\mathbb{Q}}$ with $ch(\zeta)|X_1 = [Z]$ if and only if the crystalline class $[Z]_{crys}$ lies in $F^r H^{2r}_{DR}(X/R)$ under the identification $H^*(X_1/W)_{crys} \cong H^*_{DR}(X/R)$.
- What the theorem does not say in case $R = W(k)$:
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- What the theorem says in the case $R = \mathbb{C}[[t]]$:
  If the Kunneth projectors are algebraic on $X_\eta \times X_\eta$, then "Hodgeness" of the horizontal lifting of $[Z]_{DR}$ implies that there exists a cycle Z' such that $[Z]_{DR} = [Z']_{DR}$ and Z' lifts in the above sense.

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The Motivic Complex in char. 0

- $R = k[[t]]$, $\mathbb{Q} \subset k$. $\mathbb{Z}(r)_{X_1}$ complex of Zariski sheaves calculating motivic cohomology. (e.g. shifted higher chow complex)
- $\mathbb{Z}(r)_{X_1}$ supported in $[-\infty, r]$ and $\mathcal{H}^r(\mathbb{Z}(r)_{X_1}) = \mathcal{K}_r^M$ (Milnor K-sheaf generated by symbols).
- We define $\mathcal{Z}(r)_{X_n}$ via the pullback

$$\begin{array}{ccc} \mathbb{Z}(r)_{X_n} & \longrightarrow & \mathbb{Z}(r)_{X_1} \\ \downarrow & & \downarrow \\ \mathcal{K}^M_{r,X_n}[-r] & \longrightarrow & \mathcal{K}^M_{r,X_1}[-r] \end{array}$$

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The Motivic Complex in char. 0 (cont)

- $A_\bullet = \Gamma(U, \mathcal{O}_{X_\bullet})$.
- Pro-isomorphism

$$K_*(A_\bullet, A_1) \cong \ker(K^M(A_\bullet) \to K^M(A_1)).$$

  - Goodwillie's theorem

$$K_{i+1}(A_n, A_1) \cong HC_i(A_n, A_1).$$

  - Cyclic homology is known

$$HC_i(A_n) \cong \Omega^i_{A_n}/B^i_{X_n} \oplus Z^{i-2}_{A_n}/B^{i-2}_{A_n} \oplus Z^{i-4}_{A_n}/B^{i-4}_{A_n} \cdots$$

  - Terms Z^{i-2k}/B^{i-2k} are independent of n (Poincaré lemma) and die in inverse limit

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The Motivic Complex in char. 0 (cont)

- $A_\bullet = \Gamma(U, \mathcal{O}_{X_\bullet})$.
- Pro-isomorphism

$$K_*(A_\bullet, A_1) \cong \ker(K^M(A_\bullet) \to K^M(A_1)).$$

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The Motivic Complex in Mixed Characteristic

- Idea: Codim. r cycle on X_1 defines class in $H^{2r}_{DR}(X/W)$. Want to measure obstruction to this class lying in F^r.

$$\mathbb{Z}(r)_{X_\bullet} \stackrel{?}{=} \mathrm{Cone}(\mathbb{Z}_{X_1}(r) \stackrel{??}{\to} \Omega^*_{X_\bullet/W_\bullet}/F^r)$$

- Will assume $r < p$

$$p(r)\Omega^*_{X_\bullet} : p^r\mathcal{O}_{X_\bullet} \xrightarrow{d} p^{r-1}\Omega^1_{X_\bullet} \to \cdots \to \Omega^r \to \Omega^{r+1} \to \cdots$$

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- $\Omega^*_{D_\bullet} := \Omega^*_{Z_\bullet} \otimes \mathcal{O}_{D_\bullet}$. Special relation $d\gamma_n(x) = \gamma_{n-1}(x)dx$.

$$\begin{array}{ccc} \Omega^*_{X_\bullet} & \xleftarrow{\simeq} & \Omega^*_{D_\bullet} \\ & & \downarrow \simeq \\ & & W_\bullet \Omega^*_{X_1} \end{array}$$

- $J_\bullet \subset \mathcal{O}_{D_\bullet}$ ideal of $X_\bullet$, $I_\bullet = (J_\bullet, p)$ ideal of X_1

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The p-adic Motivic Complex; Beilinson's definition

$$\mathbb{Z}_{X_\bullet}(r) := \mathrm{Cone}\Big(I(r)\Omega^*_{D_\bullet} \oplus \Omega^{\geq r}_{X_\bullet} \oplus \mathbb{Z}_{X_1}(r) \xrightarrow{\phi} p(r)\Omega^*_{X_\bullet} \oplus q(r)W\Omega^*_{X_1}\Big)$$

$$\phi = \begin{pmatrix} a & \phi_{12} & 0 \\ b & 0 & \phi_{23} \end{pmatrix}$$

Natural inclusion of complexes

$$\phi_{12} : \Omega^{\geq r}_{X_\bullet} \to p(r)\Omega^*_{X_\bullet}$$

$d\log$ map for de Rham Witt:

$$\phi_{23} : \mathbb{Z}(r)_{X_1} \to \mathcal{K}_{r,X_1}[-r] \xrightarrow{d\log} W\Omega^r_{X_1,\log}[-r] \to q(r)W\Omega^*_{X_1}.$$

Teichmuller map $\mathcal{O}^\times_{X_1} \to (W\mathcal{O}_{X_1})^\times; x \mapsto [x]$.

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Comments on the proof; mixed characteristic case

- $(\mathcal{K}/p)_{X,s}$ étale sheaf of K-groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients.
- $K =$ quotient field(W), $j : X_K \hookrightarrow X$, $i : X_1 \hookrightarrow X$ (small cheat: must adjoin p-root of 1 to W)

$$\mathfrak{V}_X(r) = \text{cone}\Big(\tau_{\leq r} Rj_*\mathbb{Z}/p\mathbb{Z}(r) \xrightarrow{res} i_*\Omega^{r-1}_{X_0,\log}[-r]\Big)[-1]$$

- For example $\mathfrak{V}_X(1) \cong \mathbb{G}_{m,X} \otimes^L \mathbb{Z}/p\mathbb{Z}[-1]$.

Theorem

Unique isomorphism of étale sheaves on X_1

$$i^*(\mathcal{K}/p)_{X,s} \xrightarrow{\cong} \bigoplus_{r \leq s} i^*\mathcal{H}^{2r-s}(\mathfrak{V}_X(r)).$$

compatible with symbols and cup product with the Bott map.

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Hodge-like conjectures in char. 0

Conjecture (Infinitesimal Hodge *Conjecture*)

$x = [Z]_{DR}$. Assume horizontal lift $\tilde{x} \in F^r H^{2r}_{DR}$. Then there exists an algebraic cycle $\mathcal{Z}$ on X such that $\tilde{x} = [\mathcal{Z}]_{DR}$.

Conjecture (Grothendieck Variational Hodge Conjecture)

$$X \xrightarrow{f} S \to \operatorname{Spec} \mathbb{C}$$

f smooth, projective, S quasi-projective, smooth. $s \in S$ a point; $\sigma \in H^{2r}_{DR}(X)$. Assume $\sigma|_{X_s}$ is the class of an algebraic cycle on X_s. Then there exists a class $\xi \in K_0(X)_{\mathbb{Q}}$ such that $[ch(\xi)]_{DR}|_{X_s} = \sigma|_{X_s}$.

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Hodge-like conjectures II

Theorem

The variational Hodge conjecture is equivalent to the infinitesimal Hodge conjecture.

K-cohomology in char. 0

- $CH^r(?) = H^r(?, \mathcal{K}_r^M)$.
- $X \to S = \text{Spf}\ \overline{\mathbb{Q}}[[t]]$ smooth projective, X a formal scheme.
- X local ringed space; can define Milnor K-sheaves $K^M_{r,X}$.
- Can prove (?)

$$K^M_{r,X} \cong \varprojlim_n K^M_{r,X_n}.$$

- Infinitesimal structure of Milnor K-sheaves:

$$0 \to \Omega^{r-1}_{X_1} \xrightarrow{b} K^M_{r,X_n} \to K^M_{r,X_{n-1}} \to 0$$

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K-cohomology in char. 0

- $CH^r(?) = H^r(?, \mathcal{K}_r^M)$.
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X smooth projective formal scheme as above. Are elements in $H^r(X, \mathcal{K}_r^M)$ given by cycles?

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