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Dear Masha,

I agree with you that the issue of hypergeometric kappas with complex alphas and betas is not particularly important and can be left as a remark. Having said that, I do want to check for myself that the assertion is true. For this we need to look at deformation of parameters. This is a bit confusing because we want to fix the singular points (poles), but we want the local exponents to move. Let R be a $\mathbb{C}$-algebra (commutative) and fix $x_0 \in \mathrm{Spec}\,(R)$ a closed $\mathbb{C}$-point. Let $L_R = L = \sum q_{n-i}(t)D^i = \sum t^j p_j(D) \in \mathcal{D} \otimes_{\mathbb{C}} R$ be a differential operator with coefficients in R and we consider the family of differential operators L_x as x runs through the closed points of $\mathrm{Spec}\,(R)$. We assume $q_0(t) \in \mathbb{C}[t, t^{-1}]$ is independent of R which has as effect that varying the closed point x does not vary the poles of L_x.

Example 1. *[Hypergeometric family] Consider the case of hypergeometrics of degree (n, n). In this case*

$$L = \prod_{i=1}^{n}(D - \alpha_i^0) - t\prod_{j=1}^{n}(D - \beta_j^0) = (1-t)D^n - \sum_{i=1}^{n}(\alpha_i^0 - t\beta_i^0)D^{n-1} + \cdots$$

If we replace constants α_i^0 and β_j^0 with variable $\alpha_i, \beta_j \in R$ we obtain a family $L_R = \prod(D - \alpha_i) - t\prod(D - \beta_j) = (1-t)D^n + \cdots$. Note that the pole set $\{0, 1, \infty\}$ does not move.

We next consider local exponents. Let $I(s) = p_0(s)$. Define

$$\mathcal{R} := R[s][I(s+n)^{-1}, n > 0 \in \mathbb{N}]/(I(s))$$

By definition, a *local exponent* is a $\mathbb{C}$-point $\ell : \mathcal{R} \to \mathbb{C}$. The arguments at the beginning of section 3 of the paper then imply that the formal generating series $\Phi_{\mathcal{R}}(s, t) = \sum_{n=0}^{\infty} a_n(s)t^{n+s}$ satisfying $L\Phi_{\mathcal{R}} = 0$ is defined. for a local exponent $\ell : \mathcal{R} \to \mathbb{C}$ we can define a differential operator L_ℓ over $\mathbb{C}$ (defined by the composition $l : R \to \mathcal{R} \xrightarrow{\ell} \mathbb{C}$) and a generating series of solutions and inhomogeneous solutions $\Phi_\ell(s, t) = \sum \ell(a_n(s))t^{n+\ell(s)}$. In this sense, $\mathrm{Spec}\,(\mathcal{R})$ parametrizes deformations of solutions.

One usually wants the parameter space $\mathrm{Spec}\,(\mathcal{R})$ to be smooth, say at a point ℓ.

Proposition 2. *Assuming R is smooth at the point defined by $l : R \to \mathcal{R} \xrightarrow{\ell} \mathbb{C}$, a sufficient condition for $\mathcal{R}$ to be smooth at ℓ is that the local exponent at the $\mathbb{C}$-point ℓ has multiplicity 1.*

Proof. We have $\mathrm{Spec}\,(\mathbb{R}[s][I(s+n)^{-1}, n > 0 \in \mathbb{N}] \to \mathrm{Spec}\,(R)$ with fibre the affine line with parameter s. The locus $I(s) = 0$ meets the fibre in the local exponents at ℓ. If all local multiplicities are one, this intersection is smooth, so by standard algebraic geometry $\mathcal{R}$ is smooth at ℓ. □

Example 3 (hypergeometric family (cont.))**.** *Take $\alpha_i, \beta_j \in R$ and let $L_R = \prod(D - \alpha_i) - t\prod(D - \beta_j)$. Let α_i^0, β_j^0 as in example 1 above arise by specializing using a $\mathbb{C}$-point $l : R \to \mathbb{C}$. Since L is assumed to be a hypergeometric, l will lift to a $\mathbb{C}$-point $\ell : \mathcal{R} \to \mathbb{C}$. At $t = 0$, the local exponents are defined by $\prod_1^n(s - \alpha_i)$. If the α_i^0 are distinct, it follows that $\mathcal{R}$ is smooth at ℓ.*

At this point it is convenient to change viewpoints. Rather than thinking of differential operators L with coordinates in a ring R, we think of $R \subset \mathcal{O}(\Delta)$ as a subring of the ring of analytic functions on a disk. We get in this way a family of differential operators parametrized by points $x \in \Delta$. Let $\{p_1, \ldots, p_n\} \subset \mathbb{P}^1$ be the polar locus. What this construction gives is a family of connections on $\mathbb{P}^1 - \{p_1, \ldots, p_n\}$ parametrized by Δ, i.e. a vector bundle $\mathcal{V}$ on $c) \times \Delta$ with a connection relative to Δ.

Fix a basepoint $b \in \mathbb{P}^1 - \{p_1, \ldots, p_n\}$. Let σ be a loop around p_n based at b. For a point $x \in \Delta$, we consider the monodromy along σ in $\mathcal{V}|_{(\mathbb{P}^1 - \{p_1,\ldots,p_n\})_x}$. σ acts fibrewise: $\sigma_* : \mathcal{V}_b \times \Delta \to \mathcal{V}_b \times \Delta$.

Take $p_1 = 0 \in \mathbb{P}^1$ and Let $\mathcal{R}$ be the R-algebra of local exponents at p_1. Let $\ell : \mathcal{R} \to \mathbb{C}$ be a smooth point. Take a disk $\Delta \to \mathrm{Spec}\,(R)$ as above in such a way that ℓ lies over the center of Δ. Let $\Phi_{\mathcal{R}}(s,t)$ be the generating series of solutions and inhomogeneous solutions parametrized by $\mathcal{R}$ as above. Fix a path μ from p_1 to b, and transport Φ_R along μ to b. Restricting to Δ and localizing analytically on Δ, we can suppose we have smooth families of sections $\phi_{i,\Delta},\ i = 0, 1, 2, \ldots$ on $\mathcal{V}_{b\times\Delta}$. We assume (after localizing) $\phi_{0,\Delta}$ does not vanish at any point on Δ. (More precisely, we assume $\phi_{0,\ell}$ doesn't vanish and then we shrink Δ if necessary.)

Assume $\sigma_* - Id : \mathcal{V}_b \to \mathcal{V}_b$ has everywhere rank ≤ 1, and that $(\sigma_* - Id)(\phi_0)|_{0\in\Delta} \neq 0$. Shrinking Δ, we can assume $(\sigma_* - Id)(\phi_0)$ is nowhere zero so $(\sigma_* - Id)$ has everywhere rank 1, and $\mathcal{L}$ over Δ defined by $\mathcal{L} := \mathrm{Image}(\sigma_* - Id) \subset \mathcal{V}_b$ is a line bundle. Identify $\mathcal{L} = \mathbb{C}_\Delta$ by sending $(\sigma_* - Id)(\phi_0) \mapsto 1 \times \Delta$.

Definition 4. *With notation as above, for $n < N$ we define $\kappa_n = (\sigma_* - Id)(\phi_n)$. This is a section of $\mathcal{L} = \mathbb{C}_\Delta$, i.e. a (possibly non-constant) analytic function on Δ.*

Corollary 5. *With assumptions as above, κ_n is an analytic function on Δ. In particular, it can be expanded in a Taylor series about $0 \in \Delta$.*

Corollary 6. *In the hypergeometric case (example 1), the formula*

$$\frac{1}{A(s)} = \sum_{n \geq 0} \kappa_{\rho,n}(s-\rho)^n \tag{1}$$

which is proven in the paper (Example 23) for the case when α's and β's are real, is true for any complex α's and β's, assuming no two coordinates differ by an integer.

Proof. Start with $\alpha_j^0, \beta_k^0 \in \mathbb{R}$. We consider the Taylor series coefficients. At the $\mathbb{R}$-point with coordinates (α^0, β^0) we have

$$\frac{\partial^n}{\partial s^n} \frac{1}{A(s)} = n! \kappa_{\rho,n}(s)$$

We have seen that the κ_n are analytic in a neighborhood, so we can now compute the Taylor series for both sides in the $\mathbb{R}$-variables α^0, β^0, and these $\mathbb{R}$-Taylor series will coincide. It follows that the formula (6) is valid in a $\mathbb{C}$-open set since both sides have convergent Taylor series. By analytic extension, the formula is valid on the open domain where the functions are defined. □