Dec. 18, 2005

Comments on Deligne-Goncharov

In my last lecture on the paper of Deligne-Goncharov, “Groupes fondamentaux motiviques de Tate mixte”, I got stuck on the final counting argument, and I promised a correction.

Recall one has a grouplike element

$$dch \in \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle \tag{1}$$

We have

$$dch = 1 - \zeta(2)[e_0, e_1] + O(deg. \geq 3) \tag{2}$$

and the coefficients c_I of monomials e_I in dch are the multiple zeta values. We are interested in polynomial relations between these multiple zeta values. Because dch is grouplike, any product of two coefficients is a linear combination of coefficients. It follows that polynomial relations $P(c_{I_1}, \ldots, c_{I_N}) = 0$ can be reduced to linear relations $\sum a_J c_J = 0$.

The Tannaka group for the category $MT(\mathbb{Z})$ of mixed Tate motives over Spec $(\mathbb{Z})$ with the fibre functor $M \mapsto \bigoplus_n Hom(\mathbb{Q}(n), gr^W_{-2n}M)$ was a semi-direct product $U \cdot \mathbb{G}_m$ where U was a pro-unipotent algebraic group. $Lie(U)$ was graded and free, with generators in degree $3, 5, 7, \ldots$. The motive associated to the nilpotent completion of $F(e_0, e_1) := \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ gave us a representation $\iota : U \to V$ where V was a pro-unipotent group associated (in a somewhat complicated way) to $F(e_0, e_1)$. The grouplike element dch gave a homomorphism $dch : A \to \mathbb{C}$ where A was the affine algebra of $\iota(U) \times \text{Spec}\,\mathbb{Q}[T]$. A is a graded $\mathbb{Q}$-algebra $A = \oplus_{d \geq 0} A_d$, and all the coefficients c_I of monomials e_I in dch of degree d (i.e. all the multiple zeta values at that level) lie in the vector space $dch(A_d)$.

Where I had difficulty was in bounding $\dim A_d$. Because of the known structure of $Lie(U)$, it follows that the affine algebra B for U (note of course that $B[T] \twoheadrightarrow A$) is a symmetric algebra with generators in degrees $3, 5, 7, \ldots$. Writing $f(t) = t^3/(1-t^2)$, the Poincaré series for B is

$$1 + f(t) + f(t)^2 + f(t)^3 + \ldots = \frac{1}{1 - f(t)} = \frac{1 - t^2}{1 - t^2 - t^3} \tag{3}$$

Finally, T has graded degree 2 ($dch(T) = \pi^2$) so the Poincaré series for $B[T]$ is $1/(1 - t^2 - t^3)$. This gives an upper bound for the linear span (and hence, as argued above, the transcendence degree) of the multiple zeta elements at any given level.