Math 270 -Final Review
May 29, 2004

Here is a list of concepts covered, starting with Cauchy's thm.

(1) Cauchy's thm.
Cauchy's thm. in a disk, homotopies $h : U \times [0,1] \to U$, Goursat's thm. and existence of primitives
(2) Cauchy's integral formula
Statement, formulas for derivatives.
(3) Taylor series
convergence
(4) Liouville's thm.
(5) analytic continuation
(6) Meromorphic functions
poles, order of poles
(7) Residues
Using residues to calculate integrals.
(8) Conformal mappings
angles
(9) $\widehat{\mathbb{C}}$
meromorphic functions
(10) Linear fractional transformations
cross-ratio, "circles" carried to "circles", $z_1, z_2, z_3 \to w_1, w_2, w_3$, $PGL_2(\mathbb{C})$
(11) Riemann Mapping Theorem
(12) Harmonic functions
conjugate harmonic functions, perpindicular level sets, mean value thm. for harmonic functions, Dirichlet problem
(13) Riemann Zeta function $\zeta(s)$
definition, infinite product expression, Fourier transform, Poisson summation, $e^{-\pi z^2}$, theta function $\vartheta(t)$, $\Gamma(s)$, $\xi(s)$, analytic extension and functional equation.

Here are some typical problems involving calculations and focusing on material after the midterm:

(1) Calculate the integrals

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4+1}; \quad \int_{-\infty}^{\infty} \frac{e^{ax} dx}{e^x+1}, 0 < a < 1.$$

(2) Discuss the conformal behavior of the function $f(z) = z^2 - z$ in the disk $D_1(0)$. Analyse what happens to angles at any non-conformal points.
(3) Define a linear fractional transformation carrying $1, i, \infty \to 0, -i, 2$. What is the cross-ratio of this transformation?
(4) Explicit a linear fractional transformation carrying the circle $|z - 1| = 1$ to the circle $|z - i| = 1/2$.
(5) Define the vertical strip $\mathcal{S} := \{z \mid Im\ z > 0,\ -\pi < Re\ z < \pi\}$. Show using the function e^{iz} that $\mathcal{S}$ is conformally equivalent to $D_1^*(0) - \{-1 < z < 0\}$.
(6) Show the function $x^3 - 3xy^2 - x^2 + y^2$ is harmonic. Explicit the conjugate harmonic function.
(7) Do all the exercises and answer all the parenthetical questions in the pdf file for the lectures on the theta function and the zeta function.

(1) (a) State the Cauchy Riemann equations.
(b) Prove the function $f(z) = \bar{z}$ (complex conjugate) is not analytic.
(c) Let $u(x, y) = \mathrm{Re}(f(z))$ be the real part of an analytic function. Show u satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(2) let $f(z)$ be analytic on a neighborhood of the disk with boundary circle γ.
(a) State the Cauchy integral formula for the value $f(z)$ at a point z in the interior of the disk.
(b) Differentiate this formula with respect to z in order to deduce a formula for $f'(z)$.
(c) Calculate

$$\int_{|z-1|=2} \frac{e^{z^3} dz}{(z-1)^2}.$$

(3) (a) Let $U \subset \mathbb{C}$ be an open set. Vaguely speaking, a differentiable mapping $f : U \to \mathbb{C}$ is conformal if it preserves angles. If f is analytic and $f'(z)$ is never 0, show that f is conformal.
(b) Give an example of an analytic function which is not conformal.

(4) (a) Define the extended complex plane $\widehat{\mathbb{C}}$. What is $\widehat{\mathbb{C}}$ as a set? What are the neighborhoods of ∞? When is a function f with domain and range in $\widehat{\mathbb{C}}$ meromorphic?
(b) Define linear fractional transformations. Show a linear fractional transformation is a meromorphic mapping $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
(c) Compute the derivative of a linear fractional transformation and use your computation to show that linear fractional transformations are conformal except possibly at ∞ and the point mapping to ∞. Define a notion of angle at ∞ so linear fractional transformations are conformal at every point of $\widehat{\mathbb{C}}$.

(5) (a) Define the notion of pole and residue for an analytic function on $D_r^*(z_0)$.

(b) Compute the following residues:

$$(i)\ \frac{1}{(z-3)(z-1)} \text{ at } z=1; \quad (ii)\ \frac{z}{\cos z - 1} \text{ at } z=0;$$
$$(iii)\ e^{a(z-\frac{1}{z})} \text{ at } z=0.$$

(6) (a) State the residue theorem.
(b) Use residues to compute the following integrals:

$$(i)\ \int_{|z|=\frac{1}{2}} \frac{\cos(e^z)}{z(1-z)^3}; \quad (ii) \int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^4}; \quad (iii)\ \int_0^{\infty} \frac{x^{1/2} dx}{x^2+1}.$$

(7) Write an essay on our study of the Riemann zeta function. Define the key objects of study. What were the main results we proved? What were the main techniques? The main auxiliary functions we used?