A NOTE ON HODGE STRUCTURES ASSOCIATED TO GRAPHS

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1. Introduction

Feynman amplitudes, which play a central role in perturbative quantum field theory, are algebro-geometric periods associated to graphs. These periods have been investigated by Broadhurst and Kreimer (see [4] and the references cited there) and shown for many special graphs to be sums of multiple zeta values. On the other hand, Belkale and Brosnan have shown [2] that the related graph motives are not in general mixed Tate. The motive associated to a graph Γ is the motive of the graph hypersurface X_Γ [3]. If the cohomology of X_Γ were mixed Tate, the function $p \mapsto \#X(\mathbb{F}_p)$ would be a polynomial in p. In [2] it is shown that this function can be quite general. In particular it is not always a polynomial in p (not even if one omits a finite set of p).

The purpose of this note is to consider the Hodge structure associated to the Betti cohomology of X_Γ. Our main tool is another variety Λ_Γ which sits as a birational cover $f : \Lambda_\Gamma \to X_\Gamma$. The variety Λ_Γ has mixed Tate cohomology. As a consequence we show

Theorem 1.1. *Let X_Γ be the graph hypersurface associated to a graph Γ. Let $p \geq 0$ be an integer. Let $W.H^p(X_\Gamma, \mathbb{Q})$ be the weight filtration on the Hodge structure. Because X_Γ is proper, it is known that $H^p(X_\Gamma, \mathbb{Q}) = W_p H^p(X_\Gamma, \mathbb{Q})$, i.e. the Hodge structure on H^p has weights $\leq p$. Then*

$$H^p(X_\Gamma, \mathbb{Q})/W_{p-1} = \begin{cases} 0 & p = 2s+1 \\ \bigoplus \mathbb{Q}(s) & p = 2s. \end{cases}$$

I.e. the quotient of H^p of weight p is (pure) Tate.

Corollary 1.2. *Let $X_{\Gamma,smooth} \subset X_\Gamma$ be the open subvariety of smooth points. Then the image of the restriction map*

$$H^p(X_\Gamma, \mathbb{Q}) \to H^p(X_{\Gamma,smooth}, \mathbb{Q})$$

is a pure Tate Hodge structure of weight p.

Because Λ_Γ is mixed Tate, the function $p \mapsto \#\Lambda(\mathbb{F}_p)$ is a polynomial in p for almost all p. We can formulate this as follows. Recall [3] that $X_\Gamma : \Psi_\Gamma = 0$ where $\Psi_\Gamma = \det M_\Gamma$ is the determinant of a symmetric matrix. To a point $x \in X_\Gamma(\mathbb{F}_p)$ we associate a weight $w(x) := 1 + p + \ldots + p^{a-1}$ where a is the corank of $M_\Gamma(x)$. For x general, the corank is 1 and $w(x) = 1$.

Theorem 1.3. *The function*

$$p \mapsto \sum_{x \in X_\Gamma(\mathbb{F}_p)} w(x)$$

is a polynomial in p outside of a finite set of p.

The crucial point in the proof of theorem 1.1 is the fact that the graph hypersurface $X_\Gamma : \det(\sum A_e \mathcal{Q}_e)$ where the $\mathcal{Q}_e$ are rank 1 symmetric matrices. Using the topology of the links associated to the stratification of X_Γ according to the rank of $\sum A_e \mathcal{Q}_e$, it should be possible to get further information about the spectral sequence for $f_\Gamma : \Lambda_\Gamma \to X_\Gamma$. In particular, one may hope to better understand $\ker f_\Gamma^*$.

This work grew out of an attempt to understand a construction of H. Esnault. Unfortunately, time did not permit us to work together on this, but I am endebted to her and to D. Kreimer for many helpful conversations.

2. Λ_Γ

Let Γ be a graph. Write $H = H_1(\Gamma, \mathbb{Q})$ and fix an identification $H \cong \mathbb{Q}^r$. I assume the loop number $r \geq 1$. Let $E = E(\Gamma)$ (resp. $V = V(\Gamma)$) be the edges (resp. vertices) of Γ, and write $n = \#E$. We have $H \subset \mathbb{Q}^E$. We identify an edge e with a functional $e^\vee$ on $\mathbb{Q}^E$ which we can restrict to H. The collection $\{e^\vee|_H\}$ (or, more correctly, the zeroes of these functionals) define a configuration of hyperplanes in $\mathbb{P}^{r-1} = \mathbb{P}(H)$. The square $(e^\vee|_H)^2$ defines a rank 1 quadratic form on H. Concretely, $e^\vee|_H$ (resp. $(e^\vee|_H)^2$) corresponds to a row vector (resp. symmetric matrix)

$$w_e = (w_{e,1}, \ldots, w_{e,r}); \quad Q_e = {}^t w_e \cdot w_e = (w_{e,i} w_{e,j}). \tag{2.1}$$

The linear transformation $\mathbb{Q}^r \to \mathbb{Q}^r$ associated to Q_e is $\beta \mapsto (w_e \cdot \beta)\, {}^t w_e$.

To a vector $a = \sum a_e e \in \mathbb{Q}^E$ we can associate the symmetric matrix $Q_a := \sum_E a_e Q_e$. We define

$$\mathbb{P}^{n-1} \times \mathbb{P}^{r-1} \supset \Lambda := \{(a, \beta) \mid Q_a(\beta) = 0\}. \tag{2.2}$$

Concretely, Λ is cut out by r equations (the zeroes of r sections of $\mathcal{O}_{\mathbb{P}^{n-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)$)

$$0 = \sum_{j=1}^{r} \sum_{E} a_e w_{e,j} \beta_j w_{e,i}; \; i = 1, \ldots, r. \tag{2.3}$$

In vectors, this becomes

$$\Lambda : \sum_{E} a_e (w_e \cdot \beta) w_e = 0. \tag{2.4}$$

Definition 2.1. *The graph polynomial* $\Psi_\Gamma \in \Gamma(\mathbb{P}^{n-1}, \mathcal{O}(r))$ *is defined by the determinant* $\det(Q_a)$.

Because the w_e span H, Ψ_Γ is not identically zero. Also, the definition of Ψ uses only the configuration $H \subset \mathbb{Q}^n$. We do not need a graph to define it. We write $X = X_\Gamma : \Psi_\Gamma = 0$. Note that by (2.2), $\Lambda \subset X \times \mathbb{P}^{r-1}$.

Proposition 2.2. *(i) There exist coherent sheaves* $\mathcal{E}$ *on* X *and* $\mathcal{F}$ *on* $\mathbb{P}^{r-1}$ *such that* $\Lambda \cong Proj(Sym(\mathcal{E})) \cong Proj(Sym(\mathcal{F}))$.
(ii) Λ *is a reduced, irreducible variety of dimension* $n-2$ *which is a complete intersection of codim.* r *in* $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$. *The projection* $p : \Lambda \to X_\Gamma$ *is birational.*

Proof. Define $\mathcal{E}$ by the presentation

$$H \otimes_{\mathbb{Q}} \mathcal{O}_X \xrightarrow{\mathcal{Q}} H^\vee \otimes_{\mathbb{Q}} \mathcal{O}_X(1) \to \mathcal{E} \to 0. \tag{2.5}$$

Here $\mathcal{Q}$ acts on the fibre over a point $a \in X$ via Q_a.

For $\mathcal{F}$, the map $a \mapsto \sum_E a_e (w_e \cdot \beta) w_e$ dualizes to a presentation

$$\mathcal{O}^r_{\mathbb{P}^{r-1}} \to \mathcal{O}^n_{\mathbb{P}^{r-1}}(1) \to \mathcal{F} \to 0. \tag{2.6}$$

The fibre $\mathcal{F}_\beta$ is the quotient of $\mathbb{Q}^{n,\vee}$ modulo the space of functionals of the form $a \mapsto \sum_e a_e (\beta \cdot w_e)(\gamma \cdot w_e)$ for $\gamma \in \mathbb{Q}^r$. We have $\dim \mathcal{F}_\beta = n - r + \varepsilon$, where ε is the codimension in $\mathbb{Q}^r$ of the span of $\{w_e \mid (w_e \cdot \beta) \neq 0\}$. Since the w_e span $\mathbb{Q}^r$, it follows that for β general, we have $\varepsilon = 0$ so $\Lambda = \mathrm{Proj}(Sym(\mathcal{F}))$ has dimension $r - 1 + n - r - 1 = n - 2$. Finally, since the support of $\mathcal{E}$ is all of X_Γ, it follows from $\dim \mathrm{Proj}(Sym(\mathcal{E})) = n - 2 = \dim X$ that the fibre of $\mathcal{E}$ over a general point of X is a line, so $\Lambda \to X$ is birational. □

Lemma 2.3. *Let* V *be a variety. Let* H^*_c *denote betti cohomology with compact supports. Assume* V *admits a finite stratification* $V = \coprod V_i$ *by locally closed sets such that* $H^*_c(V_i)$ *is mixed Tate for all* i. *Then* $H^*(V)$ *is mixed Tate.*

Proof. We have a well-defined weight filtration, and the functor $H_{betti} \mapsto gr^W H_{betti}$ is exact on the category of Hodge structures. We apply this functor to the spectral sequence which relates $H_c^*(V_i)$ to $H_c^*(V)$ and deduce a spectral sequence converging to $gr^W H_c^*(V)$ with initial terms direct sums of Tate Hodge structures $\mathbb{Q}(p)$. Since extensions of $\mathbb{Q}(p)$ by $\mathbb{Q}(p)$ are all split, it follows that $gr^W H_c^*(V) = \bigoplus \mathbb{Q}(p_i)$ so by definition $H_c^*(V)$ is mixed Tate. □

Proposition 2.4. *The betti cohomology $H^*(\Lambda)$ is mixed Tate.*

Proof. Let ε be as in the proof of proposition 2.2. We write $\varepsilon(\beta)$ to indicate the dependence on $\beta \in \mathbb{P}^{r-1}$. It is clear that $T^m := \{\beta \mid \varepsilon(\beta) \geq m\}$ is closed in $\mathbb{P}^{r-1}$ and $T^{m+1} \subset T^m$. The sets $S^m := T^m - T^{m+1}$ form a stratification of $\mathbb{P}^{r-1}$ by locally closed sets. The fibres of $\mathcal{F}$ over S^m have constant rank, so $\mathcal{F}|_{S^m}$ is a vector bundle and $\Lambda|_{S^m}$ is a projective bundle. It will suffice by the lemma to show $H_c^*(\Lambda|_{S^m})$ is mixed Tate, and by the projective bundle formula this will follow if we show $H_c^*(S^m)$ is mixed Tate.

The set T^m can be described as follows. Let $Z \subset 2^E$ be the set of all subsets $z \subset E$ such that the span of $w_e,\ e \in z$ has codimension $< m$ in $\mathbb{Q}^r$. Then T^m is the set of β such that for each $z \in Z$, $(\beta \cdot w_e) = 0$ for at least one $e \in z$. Said another way, for any subset W of edges containing at least one edge from each $z \in Z$ let $L_W \subset \mathbb{P}^{r-1}$ be the set of those β such that $(\beta \cdot w_e) = 0$ for all $e \in W$. It follows that $T^m = \bigcup L_W$ is the union of the L_w. Since the cohomology of a union of linear spaces is mixed Tate, we see that $H^*(T^m)$ is mixed Tate. From the long exact sequence relating the cohomologies of T^m, T^{m+1} to the compactly supported cohomology of S^m we deduce that $H_c^*(S^m)$ is mixed Tate as well. □

Although we do not need it to prove our result, it is interesting to look more closely at the geometry of Λ and how it relates to the combinatorics of the graph. We consider partitions $E(\Gamma) = E' \amalg E''$. Let $\Gamma', \Gamma'' \subset \Gamma$ be the unions of the corresponding sets of edges. We say that our partition is non-trivial on loops if both $h_1(\Gamma'),\ h_1(\Gamma'') \geq 1$. It is easy to check that a partition is non-trivial on loops if and only if neither $\{w_e\}_{e \in E'}$ nor $\{w_e\}_{e \in E''}$ span $\mathbb{Q}^r$.

Proposition 2.5. *The fibre of Λ over a point $\beta \in \mathbb{P}^{r-1}$ has dimension $> n - r - 1$ if and only if there exists a partition $E = E' \amalg E''$ which is non-trivial on loops such that $\beta \perp w_e$ for all $e \in E'$.*

Proof. Given β, we may take $E'' = \{e \mid (w_e \cdot \beta) \neq 0\}$. The assertion is now straightforward from the definition of ε in the proof of proposition 2.2. □

Remark 2.6. *(i)Given a partition $E = E' \amalg E''$ which is non-trivial on loops, we may define linear spaces $L' = \{\beta \mid \beta \perp w_e,\ \forall e \in E'\}$ and (analogously) L''. Then L', L'' are non-empty and disjoint. The fibre dimension of Λ is $> n - r - 1$ for $\beta \in L' \amalg L''$.*
(ii) If the graph Γ admits partitions $E = E' \amalg E''$ which are non-trivial on loops, the variety Λ will be singular. Indeed, if we differentiate the vector equation $\sum a_e(\beta \cdot w_e)w_e$ with respect to a_e (resp. β_i) we obtain $(\beta \cdot w_e)w_e$ (resp. $\sum_E a_e w_{e,i} w_e$). Suppose $\beta \perp w_e,\ \forall e \in E'$, and take $a_e = 0, e \in E'$. Then the span of these vectors is the span of $w_e,\ e \in E''$ and is strictly contained in $\mathbb{Q}^r$. But the value of the equation itself does not depend on a_e for $e \in E'$, so we get points on Λ in this way where the jacobian matrix has less than maximal rank.

In fact, the structure of the singularities of Λ is more complicated because it may happen that there is a subset $F \subset E'$ such that $\{w_e \mid e \in E'' \amalg F\}$ still does not span $\mathbb{Q}^r$. In this case, it suffices to take $a_e = 0$ for $e \in E' - F$.

Example 2.7 (Wheel and spoke graphs with 3 and 4 edges)**.** *(i) The wheel with 3 spokes graph Γ_3 has 4 vertices $1, 2, 3, 4$ and 6 edges*

$$\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 4\}, \{2, 4\}, \{3, 4\}.$$

It has 3 loops, but it is easy to check that there are no partitions of the edges which are non-trivial on loops. It follows from proposition 2.5 that in this case Λ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. In particular it is smooth.
(ii) The wheel with 4 spokes Γ_4 has 5 vertices, 4 loops, and 8 edges:

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}.$$

With the aid of a computer, one can show That $\Lambda \to \mathbb{P}^3$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}^3 - 4$ points. Over the 4 missing points, the fibre jumps to $\mathbb{P}^4$.

We return to the case of a general graph Γ. It is probably possible to calculate $gr^W H^*(\Lambda_\Gamma)$ completely. The key point is the following easy lemma:

Lemma 2.8. *Let $f : \Lambda \to \mathbb{P}^{r-1}$ be the projection. The sheaves $R^a f_* \mathbb{Q}_\Lambda$ are zero for a odd. For $a = 2b$ even, we have*

$$R^a f_* \mathbb{Q}_\Lambda \cong \mathbb{Q}(-b)|_{S_b} \tag{2.7}$$

where $S_b \subset \mathbb{P}^{r-1}$ is the closed set where the fibre dimension is $\geq b$. In particular, $S_b = \mathbb{P}^{r-1}$ for $b \leq n - r - 1$.

proof of lemma. Let $g : \Lambda \to \mathbb{P}^{n-1}$ be the other projection. Pullback g^* induces a map of sheaves on $\mathbb{P}^{r-1}$

$$g^* : H^a(\mathbb{P}^{n-1}, \mathbb{Q})_{\mathbb{P}^{r-1}} \to R^a f_* \mathbb{Q}_\Lambda. \tag{2.8}$$

The lemma follows from the fact that g^* is surjective with support on S_b. (Both assertions are checked fibrewise.) □

Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\mathbb{P}^{r-1}, R^q f_* \mathbb{Q}_\Lambda) \Rightarrow H^{p+q}(\Lambda, \mathbb{Q}). \tag{2.9}$$

It follows from the lemma that $E_2^{pq} = H^p(S_{q/2}, \mathbb{Q}(-q/2))$ (zero for q odd) has weights $\leq p+q$ with equality if either $p = 0$ or $q \leq 2(n-r-1)$. Since E_r is a subquotient of E_2, we get the same assertion for E_r. From the complex (computing E_{s+1}^{pq}.)

$$E_s^{p-s,q+s-1} \to E_s^{p,q} \to E_s^{p+s,q-s+1} \tag{2.10}$$

we deduce

Proposition 2.9. *For the spectral sequence* (2.9) *we have in the range* $q \leq 2(n-r-1)$ *or* $p = 0$, $q \leq 2(n-r)$ *that* $E_\infty^{pq} = \mathbb{Q}(-(p+q)/2)$ *if both* p *and* q *are even, and* $E_\infty^{pq} = (0)$ *otherwise. In particular, The pullback* $H^s(\mathbb{P}^{n-1} \times \mathbb{P}^{r-1}, \mathbb{Q}) \to H^s(\Lambda, \mathbb{Q})$ *is an isomorphism for* $s \leq 2(n-r)$.

Corollary 2.10. *Suppose* $n = 2r$. *Then* $W_{n-3}H^{n-2}(X_\Gamma, \mathbb{Q})$ *dies in* $H^{n-2}(\Gamma, \mathbb{Q})$.

3. Proof of Theorem 1.1

Let Γ be a graph with n edges, and let $f : \Lambda_\Gamma \to X_\Gamma$ be the birational map constructed in the previous section. Let $g : \widetilde{\Lambda} \to \Lambda$ be a resolution of singularities. By [5], proposition (8.2.5), the image of $H^p(X, \mathbb{Q}) \xrightarrow{g^* f^*} H^p(\widetilde{\Lambda}, \mathbb{Q})$ is identified with $H^p(X, \mathbb{Q})/W_{p-1} = gr_p^W H^p(X, \mathbb{Q})$. This image is a subquotient of $H^p(\Lambda, \mathbb{Q})$ which is mixed Tate by proposition 2.4. Hence it is (pure) Tate, proving theorem 1.1. To prove corollary 1.2, it suffices to remark that $H^p(X_{smooth}, \mathbb{Q})$ has weights $\geq p$ so the restriction map factors through $H^p(X, \mathbb{Q})/W_{p-1}$ which we know to be Tate.

Concerning the proof of theorem 1.3, $w(x)$ is the number of points in the fibre of Λ_Γ over x, so the assertion amounts to saying that $p \mapsto \#\Lambda(\mathbb{F}_p)$ is a polynomial. The necessity of excluding a finite set of primes arises because, viewing Λ as a scheme over $\operatorname{Spec} \mathbb{Z}$ with structure map $\alpha : \Lambda \to \operatorname{Spec} \mathbb{Z}$, the cohomology sheaves $R^i \alpha_* \mathbb{Q}_\ell$ are constructible sheaves away from ℓ so the specialization map from the closed fibre at p to the generic fibre is an isomorphism for almost all p. One might ask whether it is an isomorphism for all p. To investigate this one might try to show that the topology of the fibre over $\operatorname{Spec} \mathbb{F}_p$ of the sets $S^m \subset \mathbb{P}^{r-1}$ arising in the proof of proposition 2.4 did not depend in any essential way on the prime p.

Remark 3.1. *As suggested by results of* [4], *the piece of the cohomology of* X_Γ *of "physical interest", i.e. related to the Feynman amplitude period, may be mixed Tate. Is it possible that this piece maps injectively to* $H^*(\Lambda_\Gamma)$*? In the case of the wheel with* $m \geq 3$ *spokes (example 2.7 above) one knows from results in* [3] *that* $H^{2m-2}(X, \mathbb{Q})_{prim} \cong \mathbb{Q}(-2)$. *(Here the subscript "prim" means to kill the Lefschetz class* $Q(-m-1)$ *coming via pullback from* $H^{2m-2}(\mathbb{P}^{2m-1}, \mathbb{Q})$*. Note* $\mathbb{Q}(-2)$ *is independent of* m*.) This primitive class has weight* 4*. For* $m = 3$ *we have that* $4 = 2m - 2$*, so by theorem 1.1, this class survives in* $H^4(\Lambda, \mathbb{Q})$*. For* $m \geq 4$*, however, it follows from corollary 2.10 that* $\mathbb{Q}(-2)$ *dies in* $H^{2m-2}(\Lambda, \mathbb{Q})$.

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