MOTIVES ASSOCIATED TO SUMS OF GRAPHS

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1. Introduction

In quantum field theory, the path integral is interpreted perturbatively as a sum indexed by graphs. The coefficient (Feynman amplitude) associated to a graph Γ is a period associated to the motive given by the complement of a certain hypersurface X_Γ in projective space. Based on considerable numerical evidence, Broadhurst and Kreimer suggested [4] that the Feynman amplitudes should be sums of multizeta numbers. On the other hand, Belkale and Brosnan [2] showed that the motives of the X_Γ were not in general mixed Tate.

A recent paper of Aluffi and Marcolli [1] studied the images $[X_\Gamma]$ of graph hypersurfaces in the Grothendieck ring $K_0(Var_k)$ of varieties over a field k. Let $\mathbb{Z}[\mathbb{A}^1_k] \subset K_0(Var_k)$ be the subring generated by $1 = [\operatorname{Spec} k]$ and $[\mathbb{A}^1_k]$. It follows from [2] that $[X_\Gamma] \notin \mathbb{Z}[\mathbb{A}^1_k]$ for many graphs Γ.

Let $n \geq 3$ be an integer. In this note we consider a sum $S_n \in K_0(Var_k)$ of $[X_\Gamma]$ over all connected graphs Γ with n vertices, no multiple edges, and no tadpoles (edges with just one vertex). (There are some subtleties here. Each graph Γ appears with multiplicity $n!/|Aut(\Gamma)|$. For a precise definition of S_n see (5.1) below.) Our main result is

Theorem 1.1. $S_n \in \mathbb{Z}[\mathbb{A}^1_k]$.

For applications to physics, one would like a formula for sums over all graphs with a given loop order. I do not know if such a formula could be proven by these methods.

Dirk Kreimer explained to me the physical interest in considering sums of graph motives, and I learned about $K_0(Var_k)$ from correspondence with H. Esnault. Finally, the recently paper of Aluffi and Marcolli [1] provides a nice exposition of the general program.

2. Basic Definitions

Let E be a finite set, and let

$$0 \to H \to \mathbb{Q}^E \to W \to 0; \quad 0 \to W^\vee \to \mathbb{Q}^E \to H^\vee \to 0 \tag{2.1}$$

be dual exact sequences of vector spaces. For $e \in E$, let $e^\vee : \mathbb{Q}^E \to \mathbb{Q}$ be the dual functional, and let $(e^\vee)^2$ be the square, viewed as a quadratic function. By restriction, we can view this as a quadratic function either on H or on $W^\vee$. Choosing bases, we get symmetric matrices M_e and N_e. Let $A_e, e \in E$ be variables, and consider the homogeneous polynomials

$$\Psi(A) = \det(\sum A_e M_e); \quad \Psi^\vee(A) = \det(\sum A_e N_e). \tag{2.2}$$

Lemma 2.1. *$\Psi(\dots A_e, \dots) = c \prod_{e \in E} A_e \Psi^\vee(\dots A_e^{-1}, \dots)$, where $c \in k^\times$.*

Proof. This is proposition 1.6 in [3]. □

Let Γ be a graph. Write E, V for the edges and vertices of Γ. We have an exact sequence

$$0 \to H_1(\Gamma, \mathbb{Q}) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^V \to H_0(\Gamma, \mathbb{Q}) \to 0. \tag{2.3}$$

We take $H = H_1(\Gamma)$ and $W = \text{Image}(\partial)$ in (2.1). The resulting polynomials $\Psi = \Psi_\Gamma$, $\Psi^\vee = \Psi^\vee_\Gamma$ as in (2.2) are given by [3]

$$\Psi_\Gamma = \sum_{t \in T} \prod_{e \notin t} A_e; \quad \Psi^\vee_\Gamma = \sum_{t \in T} \prod_{e \in t} A_e. \tag{2.4}$$

Here T is the set of *spanning trees* in Γ.

Lemma 2.2. *Let $e \in \Gamma$ be an edge. Let Γ/e be the graph obtained from Γ by shrinking e to a point and identifying the two vertices. We do not consider Γ/e in the degenerate case when e is a loop, i.e. if the two vertices coincide. Let $\Gamma - e$ be the graph obtained from Γ by cutting e. We do not consider $\Gamma - e$ in the degenerate case when cutting e disconnects Γ or leaves an isolated vertex. Then*

$$\Psi_{\Gamma/e} = \Psi_\Gamma|_{A_e=0}; \quad \Psi_{\Gamma-e} = \frac{\partial}{\partial A_e}\Psi_\Gamma. \tag{2.5}$$

$$\Psi^\vee_{\Gamma/e} = \frac{\partial}{\partial A_e}\Psi^\vee_\Gamma; \quad \Psi^\vee_{\Gamma-e} = \Psi^\vee_\Gamma|_{A_e=0}. \tag{2.6}$$

(In the degenerate cases, the polynomials on the right in (2.5) *and* (2.6) *are zero.)*

Proof. The formulas in (2.5) are standard [3]. The formulas (2.6) follow easily using lemma 2.1. (In the case of graphs, the constant c in the lemma is 1.) □

More generally, we can consider strings of edges $e_1, \dots, e_p \in \Gamma$. If at every stage we have a nondegenerate situation we can conclude inductively

$$\Psi^\vee_{\Gamma - e_1 - \dots - e_p} = \Psi^\vee_\Gamma|_{A_{e_1} = \dots = A_{e_p} = 0} \tag{2.7}$$

In the degenerate situation, the polynomial on the right will vanish, i.e. X_Γ will contain the linear space $A_{e_1} = \cdots = A_{e_p} = 0$.

For example, let $\Gamma = e_1 \cup e_2 \cup e_3$ be a triangle, with one loop and three vertices. We get the following polynomials

$$\Psi_\Gamma = A_{e_1} + A_{e_2} + A_{e_3}; \quad \Psi_\Gamma^\vee = A_{e_1}A_{e_2} + A_{e_2}A_{e_3} + A_{e_1}A_{e_3} \tag{2.8}$$

$$\Psi_{\Gamma - e_i} = 1; \quad \Psi_{\Gamma - e_i}^\vee = A_{e_j}A_{e_k} = \Psi_\Gamma^\vee|_{A_{e_i}=0} \tag{2.9}$$

The sets $\{e_i, e_j\}$ are degenerate because cutting two edges will leave an isolated vertex.

3. The Grothendieck Group and Duality

Recall $K_0(Var_k)$ is the free abelian group on generators isomorphism classes $[X]$ of quasi-projective k-varieties and relations

$$[X] = [U] + [Y]; \quad U \overset{\text{open}}{\hookrightarrow} X,\ Y = X - U. \tag{3.1}$$

In fact, $K_0(Var_k)$ is a commutative ring with multiplication given by cartesian product of k-varieties. Let $\mathbb{Z}[\mathbb{A}_k^1] \subset K_0(Var_k)$ be the subring generated by $1 = [\mathrm{Spec}\, k]$ and $[\mathbb{A}_k^1]$. Let $\mathbb{P}_\Gamma$ be the projective space with homogeneous coordinates $A_e, e \in E$. We write $X_\Gamma : \Psi_\Gamma = 0,\ X_\Gamma^\vee : \Psi_\Gamma^\vee = 0$ for the corresponding hypersurfaces in $\mathbb{P}_\Gamma$. We are interested in the classes $[X_\Gamma], [X_\Gamma^\vee] \in K_0(Var_k)$.

Let $\Delta : \prod_{e \in E} A_e = 0$ in $\mathbb{P}_\Gamma$, and let $\mathbb{T} = \mathbb{T}_\Gamma = \mathbb{P}_\Gamma - \Delta$ be the torus. Define

$$X_\Gamma^0 = X_\Gamma \cap \mathbb{T}_\Gamma; \quad X_\Gamma^{\vee,0} = X_\Gamma^\vee \cap \mathbb{T}_\Gamma. \tag{3.2}$$

Lemma 2.1 translates into an isomorphism (Cremona transformation)

$$X_\Gamma^0 \cong X_\Gamma^{\vee,0}. \tag{3.3}$$

(In fact, this is valid more generally for the setup of (2.1) and (2.2).) We can stratify $X_\Gamma^\vee$ by intersecting with the toric stratification of $\mathbb{P}_\Gamma$ and write

$$[X_\Gamma^\vee] = \sum_{\{e_1,\ldots,e_p\} \subset E} [(X_\Gamma^\vee \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] \in K_0(Var_k) \tag{3.4}$$

where the sum is over all subsets of E, and superscript 0 means the open torus orbit where $A_e \neq 0, e \notin \{e_1, \ldots, e_p\}$. We call a subset $\{e_1, \ldots, e_p\} \subset E$ degenerate if $\{A_{e_1} = \cdots = A_{e_p} = 0\} \subset X_\Gamma^\vee$. Since $[\mathbb{G}_m] = [\mathbb{A}^1] - [pt] \in K_0(Var_k)$ we can rewrite (3.4)

$$[X_\Gamma^\vee] = \sum_{\substack{\{e_1,\ldots,e_p\} \subset E \\ \text{nondegenerate}}} [(X_\Gamma^\vee \cap \{A_{e_1} = \cdots = A_{e_p} = 0\})^0] + t \tag{3.5}$$

where $t \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var_k)$. Now using (2.7) and (3.3) we conclude

$$[X_\Gamma^\vee] = \sum_{\substack{\{e_1,\ldots,e_p\}\subset E \\ \text{nondegenerate}}} [(X^0_{\Gamma-\{e_1,\ldots,e_p\}}] + t. \tag{3.6}$$

4. Complete Graphs

Let Γ_n be the complete graph with $n \geq 3$ vertices. Vertices of Γ_n are written (j), $1 \leq j \leq n$, and edges e_{ij} with $1 \leq i < j \leq n$. We have $\partial e_{ij} = (j) - (i)$.

Proposition 4.1. *We have* $[X_{\Gamma_n}^\vee] \in \mathbb{Z}[\mathbb{A}^1_k]$.

Proof. Let $\mathbb{Q}^{n,0} \subset \mathbb{Q}^n$ be row vectors with entries which sum to 0. We have

$$0 \to H_1(\Gamma_n) \to \mathbb{Q}^E \xrightarrow{\partial} \mathbb{Q}^{n,0} \to 0. \tag{4.1}$$

In a natural way, $(\mathbb{Q}^{n,0})^\vee = \mathbb{Q}^n/\mathbb{Q}$. Take as basis of $\mathbb{Q}^n/\mathbb{Q}$ the elements $(1), \ldots, (n-1)$. As usual, we interpret the $(e_{ij}^\vee)^2$ as quadratic functions on $\mathbb{Q}^n/\mathbb{Q}$. We write N_e for the corresponding symmetric matrix.

Lemma 4.2. *The $N_{e_{ij}}$ form a basis for the space of all $(n-1)\times(n-1)$ symmetric matrices.*

Proof of lemma. The dual map $\mathbb{Q}^n/\mathbb{Q} \to \mathbb{Q}^E$ carries

$$(k) \mapsto \sum_{\mu>k} -e_{k\mu} + \sum_{\nu<k} e_{\nu k}; \quad k \leq n-1. \tag{4.2}$$

We have

$$(e_{ij}^\vee)^2 (\sum_{k=1}^{n-1} a_k \cdot (k)) = \begin{cases} a_i^2 - 2a_i a_j + a_j^2 & i < j < n \\ a_i^2 & j = n. \end{cases} \tag{4.3}$$

It follows that if $j < n$, $N_{e_{ij}}$ has -1 in positions (ij) and (ji) and $+1$ in positions $(ii), (jj)$ (resp. N_{in} has 1 in position (ii) and zeroes elsewhere). These form a basis for the symmetric $(n-1) \times (n-1)$ matrices. □

It follows from the lemma that $X_{\Gamma_n}^\vee$ is identified with the projectivized space of $(n-1) \times (n-1)$ matrices of rank $\leq n-2$. In order to compute the class in the Grothendieck group we detour momentarily into classical algebraic geometry. For a finite dimensional k-vector space U, let $\mathbb{P}(U)$ be the variety whose k-points are the lines in U. For a k-algebra R, the R-points $\operatorname{Spec} R \to \mathbb{P}(U)$ are given by pairs (L, ϕ) where L on *Spec* R is a line bundle and $\phi : L \hookrightarrow U \otimes_k R$ is a locally split embedding.

Suppoose now $U = Hom(V,W)$. We can stratify $\mathbb{P}(Hom(V,W)) = \coprod_{p>0} \mathbb{P}(Hom(V,W))^p$ according to the rank of the homomorphism. Looking at determinants of minors makes it clear that $\mathbb{P}(Hom(V,W))^{\le p}$ is closed. Let R be a local ring which is a localization of a k-algebra of finite type, and let a be an R-point of $\mathbb{P}(Hom(V,W))^p$. Choosing a lifting b of the projective point a, we have

$$0 \to \ker(b) \to V \otimes R \xrightarrow{b} W \otimes R \to \mathrm{coker}(b) \to 0, \tag{4.4}$$

and coker(b) is a finitely generated R-module of constant rank $\dim W - p$ which is therefore necessarily free.

Let $Gr(\dim V - p, V)$ and $Gr(p, W)$ denote the Grassmann varieties of subspaces of the indicated dimension in V (resp. W). On $Gr(\dim V - p, V) \times Gr(p, W)$ we have rank p bundles E, F given respectively by the pullbacks of the universal quotient on $Gr(\dim V - p, V)$ and the universal subbundle on $Gr(p, W)$. It follows from the above discussion that

$$\mathbb{P}(Hom(V,W))^p = \mathbb{P}(\mathrm{Isom}(E,F)) \subset \mathbb{P}(Hom(E,F)). \tag{4.5}$$

Suppose now that $W = V^\vee$. Write $\langle\ ,\ \rangle : V \otimes V^\vee \to k$ for the canonical bilinear form. We can identify $Hom(V, V^\vee)$ with bilinear forms on V

$$\rho : V \to V^\vee \leftrightarrow (v_1, v_2) \mapsto \langle v_1, \rho(v_2)\rangle. \tag{4.6}$$

Let $SHom(V, V^\vee) \subset Hom(V, V^\vee)$ be the subspace of ρ such that the corresponding bilinear form on V is symmetric. Equivalently, $Hom(V, V^\vee) = V^{\vee,\otimes 2}$ and $SHom(V, V^\vee) = Sym^2(V^\vee) \subset V^{\vee,\otimes 2}$.

For ρ symmetric as above, one seees easily that $\rho(V) = \ker(V)^\perp$ so there is a factorization

$$V \to V/\ker(\rho) \xrightarrow{\cong} (V/\ker(\rho))^\vee = \ker(\rho)^\perp \hookrightarrow V^\vee. \tag{4.7}$$

The isomorphism in (4.7) is also symmetric.

Fix an identification $V = k^n$ and hence $V = V^\vee$. A symmetric map is then given by a symmetric $n \times n$ matrix. On $Gr(n-p, n)$ we have the universal rank p quotient $Q = k^n \otimes \mathcal{O}_{Gr}/K$, and also the rank p perpendicular space $K^\perp$ to the universal subbundle K. Note $K^\perp \cong Q^\vee$. It follows that

$$\mathbb{P}(SHom(k^n, k^n))^p \cong \mathbb{P}(SHom(Q, Q^\vee))^p \subset \mathbb{P}(SHom(Q, Q^\vee)). \tag{4.8}$$

This is a fibre bundle over $Gr(n-p, n)$ with fibre $\mathbb{P}(Hom(k^p, k^p))^p$, the projectivized space of symmetric $p \times p$ invertible matrices.

We can now compute $[X^\vee_{\Gamma_n}]$ as follows. Write $c(n,p) = [\mathbb{P}(SHom(k^n,k^n))^p]$. We have the following relations:

$$c(n,1) = [\mathbb{P}^{n-1}]; \quad \sum_{p=1}^{n} c(n,p) = [\mathbb{P}^{\binom{n+1}{2}-1}]; \tag{4.9}$$

$$c(n,p) = [Gr(n-p,n)]\cdot c(p,p) \tag{4.10}$$

$$[X^\vee_{\Gamma_n}] = \sum_{p=1}^{n-2} c(n-1,p) \tag{4.11}$$

Here (4.10) follows from (4.8). It is easy to see that these formulas lead to an expression for $[X^\vee_{\Gamma_n}]$ as a polynomial in the $[\mathbb{P}^N]$ and $[Gr(n-p-1,n-1)]$ (though the precise form of the polynomial seems complicated). To finish the proof of the proposition, we have to show that $[Gr(a,b)] \in \mathbb{Z}[\mathbb{A}^1_k]$. Fix a splitting $k^b = k^{b-a}\oplus k^a$. Stratify $Gr(a,b) = \coprod_{p=0}^a Gr(a,b)^p$ where

$$\begin{aligned} &Gr(a,b)^p = \\ &\{V\subset k^{b-a}\oplus k^a \mid \dim(V)=a,\ \mathrm{Image}(V\to k^a) \text{ has rank } p\} = \\ &\{(X,Y,f)\mid X\subset k^{b-a},\ Y\subset k^a,\ f:Y\to X\}\end{aligned} \tag{4.12}$$

where $\dim X = a-p,\ \dim(Y)=p$. This is a fibration over $Gr(b-a-p,b-a)\times Gr(p,a)$ with fibre $\mathbb{A}^{p(b-a-p)}$. By induction, we may assume $[Gr(b-a-p,b-a)\times Gr(p,a)]\in\mathbb{Z}[\mathbb{A}^1_k]$. Since the class in the Grothendieck group of a Zariski locally trivial fibration is the class of the base times the class of the fibre, we conclude $[Gr(a,b)^p]\in\mathbb{Z}[\mathbb{A}^1_k]$, completing the proof. ∎

In fact, we will need somewhat more.

Lemma 4.3. *Let* Γ *be a graph.*
(i) Let $e_0\in\Gamma$ *be an edge. Define* $\Gamma' = \Gamma\cup\varepsilon$*, the graph obtained from* Γ *by adding an edge* ε *with* $\partial\varepsilon = \partial e_0$*. Then* $X^\vee_{\Gamma'}$ *is a cone over* $X^\vee_\Gamma$*.*
(ii) Define $\Gamma'=\Gamma\cup\varepsilon$ *where* ε *is a tadpole, i.e.* $\partial\varepsilon = 0$*. Then* $X^\vee_{\Gamma'}$ *is a cone over* $X^\vee_\Gamma$*.*

Proof. We prove (i). The proof of (ii) is similar and is left for the reader.

Let E,V be the edges and vertices of Γ. We have a diagram

$$\begin{array}{ccc} \mathbb{Q}^E & \xrightarrow{\partial} & \mathbb{Q}^V \\ \downarrow & & \| \\ \mathbb{Q}^E\oplus\mathbb{Q}\cdot\varepsilon & \xrightarrow{\partial} & \mathbb{Q}^V \end{array} \tag{4.13}$$

Dualizing and playing our usual game of interpreting edges as functionals on $\mathrm{Image}(\partial)^\vee \cong \mathbb{Q}^V/\mathbb{Q}$, we see that $\varepsilon^\vee = e_0^\vee$. Fix a basis for $\mathbb{Q}^V/\mathbb{Q}$ so the $(e^\vee)^2$ correspond to symmetric matrices M_e. We have

$$X_\Gamma^\vee : \det(\sum_E A_e M_e) = 0; \quad X_{\Gamma'}^\vee : \det(A_\varepsilon M_{e_0} + \sum_E A_e M_e) = 0. \tag{4.14}$$

The second polynomial is obtained from the first by the substitution $A_{e_0} \mapsto A_{e_0} + A_\varepsilon$. Geometrically, this is a cone as claimed. □

Let Γ_N be the complete graph on $N \geq 3$ vertices. Let $\Gamma \supset \Gamma_N$ be obtained by adding r new edges (but no new vertices) to Γ_N.

Proposition 4.4. $[X_\Gamma^\vee] \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var_k)$.

Proof. Note that every pair of distinct vertices in Γ_N are connected by an edge, so the r new edges e either duplicate existing edges or are tadpoles ($\partial e = 0$). It follows from lemma 4.3 that $X_\Gamma^\vee$ is an iterated cone over $\mathbb{X}_{\Gamma_N}^\vee$. In the Grothendieck ring, the class of a cone is the sum of the vertex point with a product of the base times an affine space, so we conclude from proposition 4.1. □

5. The Main Theorem

Fix $n \geq 3$. Let Γ_n be the complete graph on n vertices. It has $\binom{n}{2}$ edges. Recall (lemma 2.2) a set $\{e_1, \ldots, e_p\} \subset \mathrm{edge}(\Gamma_n)$ is nondegenerate if cutting these edges (but leaving all vertices) does not disconnect Γ_n. (For the case $n = 3$ see (2.8) and (2.9).) Define

$$S_n := \sum_{\substack{\{e_1,\ldots,e_p\}\\ \text{nondegenerate}}} [X_{\Gamma_n - \{e_1,\ldots,e_p\}}] \in K_0(Var_k). \tag{5.1}$$

Let Γ be a connected graph with n vertices and no multiple edges or tadpoles. Let $G \subset Sym(\mathrm{vert}(\Gamma))$ be the subgroup of the symmetric group on the vertices which acts on the set of edges. Then $[X_\Gamma]$ appears in S_n with multiplicity $n!/|G|$.

Theorem 5.1. $S_n \in \mathbb{Z}[\mathbb{A}_k^1] \subset K_0(Var_k)$.

Proof. It follows from (3.6) and proposition 4.1 that

$$\sum_{\substack{\{e_1,\ldots,e_p\}\\ \text{nondegenerate}}} [X^0_{\Gamma_n - \{e_1,\ldots,e_p\}}] \in \mathbb{Z}[\mathbb{A}_k^1]. \tag{5.2}$$

Write $\vec{e} = \{e_1, \ldots, e_p\}$ and let $\vec{f} = \{f_1, \ldots, f_q\}$ be another subset of edges. We will say the pair $\{\vec{e}, \vec{f}\}$ is nondegenerate if $\vec{e}$ is nondegenerate in the above sense, and if further $\vec{e} \cap \vec{f} = \emptyset$ and the edges of $\vec{f}$ do not support a loop. For $\{\vec{e},\vec{f}\}$ nondegenerate, write $(\Gamma_n-\vec{e})/\vec{f}$ for the graph obtained from Γ_n by removing the edges in $\vec{e}$ and then contracting the edges in $\vec{f}$. If we fix a nondegenerate $\vec{e}$, we have

$$\sum_{\substack{\vec{f} \\ \{\vec{e},\vec{f}\}\text{ nondeg.}}} [X^0_{(\Gamma_n-\vec{e})/\vec{f}}] + t = [X_{\Gamma_n-\vec{e}}]. \tag{5.3}$$

Here $t\in\mathbb{Z}[\mathbb{A}^1]$ accounts for the $\vec{f}$ which support a loop. These give rise to degenerate edges in $X_{\Gamma_n-\vec{e}}$ which are linear spaces and hence have classes in $\mathbb{Z}[\mathbb{A}^1]$. Summing now over both $\vec{e}$ and $\vec{f}$, we conclude

$$S_n \equiv \sum_{\substack{\{\vec{e},\vec{f}\} \\ \text{nondegen.}}} [X^0_{(\Gamma_n-\vec{e})/\vec{f}}] \mod \mathbb{Z}[\mathbb{A}^1]. \tag{5.4}$$

Note that if $\vec{e},\vec{f}$ are disjoint and $\vec{f}$ does not support a loop, then $\vec{e}$ is nondegenerate in Γ_n if and only if it is nondegenerate in $\Gamma_n/\vec{f}$. This means we can rewrite (5.4)

$$S_n \equiv \sum_{\vec{f}} \sum_{\substack{\vec{e}\subset\Gamma_n/\vec{f} \\ \text{nondegen.}}} [X^0_{(\Gamma_n/\vec{f})-\vec{e}}]. \tag{5.5}$$

Let $\vec{f}=\{f_1,\dots,f_q\}$ and assume it does not support a loop. Then $\Gamma_n/\vec{f}$ has $n-q$ vertices, and every pair of distinct vertices is connected by at least one edge. This means we may embed $\Gamma_{n-q}\subset\Gamma_n/\vec{f}$ and think of $\Gamma_n/\vec{f}$ as obtained from Γ_{n-q} by adding duplicate edges and tadpoles. We then apply proposition 4.4 to conclude that $[X^\vee_{\Gamma_n/\vec{f}}]\in\mathbb{Z}[\mathbb{A}^1_k]$. Now arguing as in (3.6) we conclude

$$\sum_{\substack{\vec{e}\subset\Gamma_n/\vec{f} \\ \text{nondegen.}}} [X^0_{(\Gamma_n/\vec{f})-\vec{e}}] \in \mathbb{Z}[\mathbb{A}^1_k] \tag{5.6}$$

Finally, plugging into (5.5) we get $S_n\in\mathbb{Z}[\mathbb{A}^1]$ as claimed. □

References

[1] Aluffi, P., and Marcolli, M., Feynman Motives of Banana Graphs, arXiv:0807.1690v2 [hep-th]

[2] Belkale, P., and Brosnan, P., Matroids, motives and a conjecture on Kontsevich, Duke Math. Journal, Vol. 116 (2003) 147-188.

[3] Bloch, S, Esnault, H., and Kreimer, D., On Motives Associated to Graph Polynomials, Comm. Math. Phys. **267** (2006), 181-225.

[4] Broadhurst, D.J., and Kreimer, D., Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Phys. Lett.**B 393** (1997) 403.

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