Introduction to Twistors
Amplitude Integrals via Twistors

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Introduction

Basic amplitude integral, written in parametrized form:

- Γ graph with m edges and n loops ($+$ masses, external momenta, etc.)
- Symanzik Polynomials:

  First S. polynomial:

  $S_1(A_1, \ldots, A_m)$ homog. deg. n

  Second S. polynomial:

  $S_2(A_1, \ldots, A_m; \text{ext.momenta, masses})$

  deg. $n + 1$ in A's;

  quadratic in ext. momenta and masses.

Introduction (continued)

- Volume form on $\mathbb{P}^{m-1}$

$$\Omega := \sum (-1)^{i-1} A_i dA_1 \wedge \cdots \widehat{dA_i} \cdots \wedge dA_m.$$

- Integration Chain

$$\delta = \{(a_1, \ldots, a_m) \in \mathbb{P}^{m-1}(\mathbb{R}) \mid a_i \geq 0\}$$

- Amplitude Integral

$$\mathcal{A}(\Gamma, \text{masses, ext. momen.}) := \int_\delta \frac{S_1^{m-2n-2}\Omega}{S_2^{m-2n}}$$

Special Cases

- Log divergent case: $m = 2n$.

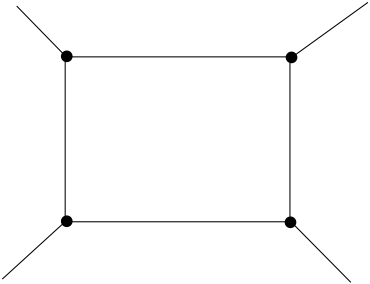
$$\mathcal{A}(\Gamma) := \int_\delta \frac{\Omega}{S_1^2}$$

- Twistor case: $m = 2n + 2$.

$$\mathcal{A}(\Gamma, \text{masses, ext. momen.}) := \int_\delta \frac{\Omega}{S_2^2}$$

The twistor amplitude depends on parameters like masses and external momenta. It is not simply an invariant of the graph.

Twistor Integral; $n = 1$.



- Basic Reference: Hodges, A., The box integrals in momentum-twistor geometry, arXiv:1004.3323v1 [hep-th].
- Bloch + Kreimer, Feynman amplitudes and Landau singularities for 1-loop graphs

Conclusion

- Γ n loops, $2n+2$ edges. Euclidean propagators $p_e, \ e \in \mathrm{Edge}(\Gamma)$. Amplitude

$$\mathcal{A}(\Gamma, q_{extern}, M) = \int_{\mathbb{R}^{4n}} \frac{d^{4n}x}{\prod_e p_e}.$$

- $\delta = \{(a_1, \ldots, a_{2n+2}) \mid a_i \geq 0\} \subset \mathbb{P}^{2n+1}(\mathbb{R})$.
- $O = \mathbb{C}^2, \ I = \mathbb{C}^{2n}, \ V = O \oplus I$. $\exists Q_e \in \bigwedge^2 V^\vee$ (interpret as $(2n+2) \times (2n+2)$ matrix) such that

$$\mathcal{A}(\Gamma, q_{extern}, M) = (const.) \int_\delta \frac{\Omega_{2n+1}}{Pfaffian(\sum a_e Q_e)^2}.$$

- We will identify Minkowski space with $Hom(O_\mathbb{R}, I_\mathbb{R}) \subset Grass(2, V)(\mathbb{R})$.
- Twistors are an exotic compactification of Minkowski space.

Disclaimer

- $Q_e \in \bigwedge^2 V^\vee$, $\dim V = 2n+2 \Rightarrow \deg\ \mathrm{Pfaff}(\sum a_e Q_e) = n+1$.
- Γ has n loops $\Rightarrow \deg S_2(\Gamma) = n+1$.
- $$(const.)\int_\delta \frac{\Omega_{2n+1}}{Pfaffian(\sum a_e Q_e)^2} = \mathcal{A}(\Gamma, q_{extern}, M) = (const.)\int_\delta \frac{\Omega_{2n+1}}{S_2^2}.$$
- Presumably, when Γ has n loops and $2n+2$ edges, we have $S_2(\Gamma, q_{extern}, M) = (const.)\mathrm{Pfaff}(\sum a_e Q_e)$. I do not prove this.

More about the Symanzik polynomials; S_1

- Definition of S_1:
  e an edge. $e^\vee : H_1(\Gamma, \mathbb{Q}) \to \mathbb{Q}$.
  $e^{\vee,2} : H_1 \to \mathbb{Q}$ rank 1 quadric $\leftrightarrow$ M_e rank 1 symmetric matrix

$$S_1 = \det(\sum_e A_e M_e).$$

- Philosophy:
  S_1 = Det of sum of rank 1 symmetric matrices $\rightsquigarrow$
  Clues about the motive of the graph hypersurface
  $X_\Gamma : S_1 = 0 \rightsquigarrow$ Information about the amplitude $\mathcal{A}(\Gamma)$.

Symanzik polynomials; S_2 with zero masses

- $S_2(\Gamma, q_{extern}, 0)$:
  $A = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ quaternions.
  $H_1(\Gamma, A) = H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} A$. $E = \mathrm{Edge}(\Gamma)$; $V = \mathrm{Vertices}(\Gamma)$

$$q_{extern} : A \to \coprod_V^0 A$$

Quaternionic configuration $W \subset \coprod_E A$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(\Gamma, A) & \longrightarrow & \coprod_E A & \longrightarrow & \coprod_V^0 A & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow q_{extern} & & \\ 0 & \longrightarrow & H_1(\Gamma, A) & \longrightarrow & W & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Symanzik polynomials; S_2 with zero masses (Cont.)

- $e \in E,\ e^{\vee} : H_1(\Gamma, A) \to A$.
  Symmetric rank 1 matrix $e^{\vee,2}$ replaced with hermitian quaternionic rank 1

$$e^{\vee}\bar{e}^{\vee} : H_1(\Gamma, A) \to \mathbb{R}.$$

- Quaternionic pfaffian $= \sqrt{\text{reduced norm}}$ for quaternionic hermitian matrices.

Theorem (Feynman amplitudes and Landau singularities for 1-loop graphs (with Dirk Kreimer))

$S_2(\Gamma, q_{extern}, 0) = quaternionic\ pfaffian(\sum_{e\in E} A_e e^{\vee}\bar{e}^{\vee})$.

$S_2(\Gamma, q_{extern}, M)$; the general case.

- $M = \{m_e\}_{e \in E}$ masses. General formula

$$S_2(\Gamma, q_{extern}, M) = S_2(\Gamma, q_{extern}, 0) - \sum_{e \in E} (m_e^2 A_e) \cdot S_1(\Gamma)$$

- Question: Can S_2 be interpreted as a Pfaffian when $M \neq 0$?

Let's do the Twist(or)

- $W \subset V$ vector spaces. $\dim W = d$. $\bigwedge^d W = \text{line} \subset \bigwedge^d V$.

$$Grass(d, V) \hookrightarrow \mathbb{P}(V) \quad \text{(Plucker embedding)}$$

- $V = W \oplus W' \rightsquigarrow U = Hom(W, W') \overset{\text{open}}{\hookrightarrow} Grass(d, V)$

$$\phi \in Hom(W, W') \mapsto \text{Graph}(\phi) = \{(w, \phi(w)) \in W \oplus W' = V\}$$

Fix bases for W and W'. $n = \dim V$

$$U = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & \phi_{11} & \cdots & \phi_{1,n-d} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \phi_{d,1} & \cdots & \phi_{d,n-d} \end{pmatrix} \right\} \cong M_{d,n-d}.$$

Twistors (cont.)

- $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{2n+2}$;
  $O = \mathbb{C}e_1 \oplus \mathbb{C}e_2$;
  $I = \mathbb{C}e_3 \oplus \cdots \oplus \mathbb{C}e_{2n+2} = \bigoplus_{i=1}^n I_i$; $I_i = \mathbb{C}e_{2i+1} \oplus \mathbb{C}e_{2i+2}$

$$\mathbb{C}^{4n} \cong \mathrm{Hom}(O, I) \hookrightarrow Grass(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V).$$

- Idea: Identify $\mathbb{C}^{4n}$ with (complexified) Minkowski space.
  $Grass(2, V)$ becomes a compactification.
- For $\alpha \in \bigwedge^2 V^\vee$, map

$$\phi \mapsto \langle (e_1 + \phi(e_1)) \wedge (e_2 + \phi(e_2)), \alpha \rangle$$

  defines a quadratic map $q_\alpha : \mathbb{C}^{2n} = \mathrm{Hom}(O, I) \to \mathbb{C}$.
- q_α not necessarily homogeneous.

The decomposable case

Lemma

Assume $\alpha = v \wedge w \in \bigwedge^2 V^\vee$. *The quadratic map* $q_\alpha : Hom(O, I) = \mathbb{C}^{4n} \to \mathbb{C}$ *has rank* ≤ 4.

Proof.

For $i, j \geq 3$

$$\langle (e_1 + \sum_{k\geq 3} x_k e_k) \wedge (e_2 + \sum_{\ell \geq 3} y_\ell e_\ell), e_i^\vee \wedge e_j^\vee \rangle = x_i y_j - x_j y_i;$$

a rank 4 quadric in x, y. Other cases are degenerate and yield lower ranks. $\square$

The decomposable case (cont.)

Lemma

Assume $\alpha = (\sum_{i=1}^{n} a_i e_{2i+1}^{\vee}) \wedge (\sum_{i=1}^{n} a_i e_{2i+2}^{\vee})$ *in* $\bigwedge^2 V^{\vee}$. *Let*

$$M_i = \begin{pmatrix} x_{2i+1} & x_{2i+2} \\ y_{2i+1} & y_{2i+2} \end{pmatrix} \in Hom(O, I_i).$$

Then

$$q_\alpha(M_1 \oplus \cdots \oplus M_n) = \det(a_1 M_1 + \cdots + a_n M_n).$$

The general case

- Inhomogeneous propagator $M = 0$, p_i, s 4-vectors

$$(p_1, \ldots, p_n) \mapsto (\sum a_i p_i + s)^2$$

$$\begin{aligned}
&\langle (e_1 + \sum_{i\geq 3} x_i e_i) \wedge (e_2 + \sum_{i \geq 3} y_i e_i), \\
&(c_1 e_1^\vee + c_2 e_2^\vee + \sum_{i\geq 1} a_i e_{2i+1}^\vee) \wedge (d_1 e_1^\vee + d_2 e_2^\vee + \sum_{i \geq 1} a_i e_{2i+2}^\vee)\rangle \\
&\qquad = \det \begin{pmatrix} \sum a_i x_{2i+1} + c_1 & \sum a_i x_{2i+2} + d_1 \\ \sum a_i x_{2i+1} + c_2 & \sum a_i x_{2i+2} + d_2 \end{pmatrix}.
\end{aligned}$$

- $M \neq 0$. Add $M^2 e_1^\vee \wedge e_2^\vee$:

$$(c_1 e_1^\vee + c_2 e_2^\vee + \sum_{i\geq 1} a_i e_{2i+1}^\vee) \wedge (d_1 e_1^\vee + d_2 e_2^\vee + \sum_{i \geq 1} a_i e_{2i+2}^\vee) +$$

$$M^2 e_1^\vee \wedge e_2^\vee \in \bigwedge^2 V^\vee \quad \text{(not decomposable if } M \neq 0).$$

The Twistor Integral

$$V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{2n+2}; \quad S = \{(v, w) \in V \times V \mid v \wedge w = 0\}$$

$$V \times V - S \xrightarrow{\rho} Grass(2, V) \xrightarrow{j} \mathbb{P}(\bigwedge^2 V).$$

$$\rho(v, w) = \text{plane spanned by } v \text{ and } w; \quad j(W) = \bigwedge^2 W \subset \bigwedge^2 V$$

The Frame Bundle

Lemma

$\rho : V \times V - S \to Grass(2, V)$ is the principal $GL_2(\mathbb{C})$-bundle (frame bundle) associated to the rank 2 vector bundle $\mathcal{W}$ on $Grass(2, V)$ which associates to a point of $Grass(2, V)$ the corresponding rank 2 subspace of V.

This simply says $\rho^{-1}(W) = \{(w_1, w_2) \mid W = \mathbb{C}w_1 \oplus \mathbb{C}w_2\}$.

The Alternating Form Q

$Q \in \bigwedge^2 V^\vee$, non-degenerate alternating form on V. Function on $V \times V$

$$(x, y) \mapsto \sum_{\mu,\nu} x_\nu Q^{\nu\mu} y_\mu = xQy; \quad R = \{(x, y) \mid xQy = 0\}.$$

Lemma

$S \subset R$. We have

$$H^i_{Betti}(V \times V - S, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 4n+1, 4n+3, 8n+4 \\ (0) & \textit{else.} \end{cases}$$

$$H^i_{Betti}(V \times V - R, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1, 4n+3, 4n+4 \\ (0) & \textit{else.} \end{cases}$$

Periods in middle dimension

- $\dim_{\mathbb{R}} V \times V = 8n + 8$. As a Hodge structure,

$$H^{4n+4}(V \times V - R, \mathbb{Q}) = \mathbb{Q}(-2n-3).$$

- $\mu := dv/Q^{2n+2}$ non-trivial in $H^{4n+4}(V \times V - R, \mathbb{C})$.
- Goal: Understand the structure of a $4n+4$-chain $\sigma \neq 0$ in $H_{4n+4}(V \times V - R, \mathbb{Q})$.
- But first...

Feynman Trick

- Scale σ so

$$\int_\sigma d^{2n+2}z \wedge d^{2n+2}w/(\sum z_\mu Q^{\mu p} w_p)^{2n+2} = \frac{(2\pi i)^{2n+3}}{\det Q} \quad (1)$$

Note this is compatible with changing Q.

- Feynman Trick is integral identity ($\delta \subset \mathbb{P}^{2n+1}(\mathbb{R})$):

$$\frac{1}{\prod_{i=1}^{2n+2} A_i} = (2n+1)! \int_\delta \frac{\Omega(a)}{(\sum a_i A_i)^{2n+2}}. \quad (2)$$

Application of Feynman Trick

Apply F.T. with

$$A_i = \sum_{\mu,p=1}^{2n+2} z_\mu Q_i^{\mu,p} w_p; \quad Q_i \text{ skew symmetric}$$

Take $Q = \sum_i a_i Q_i$. Integrate over chain σ on $V \times V - R$.

$$\int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}w}{\prod_{i=1}^{2n+2}(\sum_{\mu,p} z_\mu Q_i^{\mu,p} w_p)} \overset{\text{u.e.i.}}{=}$$

$$(2n+1)!(2\pi i)^{2n+3} \int_\delta \frac{\Omega(a)}{\text{Pfaffian}(\sum a_i Q_i)^2}. \quad (3)$$

(u.e.i. means unjustified exchange of integration.)

Amplitude Calculation

- Reinterpret $\int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}w}{\prod_{i=1}^{2n+2}(\sum_{\mu,p} z_\mu Q_i^{\mu,p} w_p)}$ as amplitude integral.
- Given graph Γ with $2n+2$ edges. Terms $\sum_{\mu,p} z_\mu Q_i^{\mu,p} w_p$ will be propagators.
- Show we can choose σ fibred over $\mathbb{R}$-Minkowski space in such a way that integral becomes familiar propagator integral over Minkowski space.

$$\sigma \subset V \times V - R \subset V \times V - S \xrightarrow{\rho} Grass(2, V) \supset \mathbb{C}^{2n} \supset \mathbb{R}^{2n}.$$

Topology

- Put an Hermitian metric $||\cdot||$ on $V = \mathbb{C}^{2n+2}$.
- Induced metric of bundle of 2-planes defines $M \subset V \times V - S$

$$M = \{(v, w) \mid ||v|| = ||w|| = 1, \langle v, w \rangle = 0\}.$$

- M is a U_2-bundle, reduction of structure of GL_2-bundle $V \times V - S$. $M \simeq V \times V - S$.
- Fibre $\rho^{-1}(w) \simeq \rho^{-1}(w) \cap M \cong U_2$; compact oriented 4-manifold.
- $Q \in \bigwedge^2 V^\vee$, $G^0 := Grass(2, V) - \{Q = 0\}$ (Q max. rank)

$$R - S = \rho^{-1}\{Q = 0\}; \quad V \times V - R \xrightarrow{\rho} G^0.$$

Topology (cont.)

- $R^4\rho_*\mathbb{Z}$ local system on $Grass(2, V) \Rightarrow R^4\rho_*\mathbb{Z} = \mathbb{Z}_{Grass}$.
- $\dim_{\mathbb{C}} Grass = 4n$, $G^0 \overset{\text{affine}}{\hookrightarrow} Grass \Rightarrow cd\ G^0 = 4n$.
- $\therefore \mathbb{Q} = H^{4n+4}(V \times V - R, \mathbb{Q}) \cong H^{4n}(G^0, \mathbb{Q})$.
- Must relate $\mathbb{R}$-Minkowski space to a non-trivial cycle in $H_{4n}(G^0, \mathbb{Q})$.

The amplitude integral again

- $q : \mathbb{R}^4 \to \mathbb{R}$; $q(x_1, \dots, x_4) = x_1^2 + \cdots + x_4^2$. Complex coordinates:

$$z_1 = x_1 + ix_2,\ z_2 = ix_3 + x_4,\ w_1 = ix_3 - x_4,\ w_2 = x_1 - ix_2;$$

$$q = z_1 w_2 - z_2 w_1.$$

$$\text{Real structure } \mathbb{R}^4 = \{(z_1, z_2, -\bar{z}_2, \bar{z}_1) \mid z_j \in \mathbb{C}\}$$

- Fix real coordinates on $H_1(\Gamma, \mathbb{R})$. Get $\mathbb{C}$-coordinates $(z_1^k, z_2^k, w_1^k, w_2^k)$ on $H_\mathbb{C} := H_1 \otimes \mathbb{C}^4$.
- For each edge e, $\exists \alpha_k = \alpha_k(e) \in \mathbb{R}$, propagator for e is

$$\begin{pmatrix} \sum_k \alpha_k z_1^k & \sum_k \alpha_k z_2^k \\ -\sum_k \alpha_k \bar{z}_2^k & \sum_k \alpha_k \bar{z}_1^k \end{pmatrix} = \Big| \sum_k \alpha_k z_1^k \Big| + \Big| \sum_k \alpha_k z_2^k \Big|.$$

Positive linear com. of propagators > 0 on $H = H_1(\Gamma, \mathbb{R}) \otimes \mathbb{R}^4$.

Twistors again

- Use coordinates z_i^k, w_i^k to identify $H_{\mathbb{C}} = H_1 \otimes \mathbb{C}^4$ with open in $Grass(2, V)$.

$$(z, w) \mapsto \begin{pmatrix} 1 & 0 & z_1^1 & z_2^1 & z_1^2 & z_2^2 & \cdots & z_1^n & z_2^n \\ 0 & 1 & w_1^1 & w_2^1 & w_2^2 & w_2^2 & \cdots & w_1^n & w_2^n \end{pmatrix}.$$

- throw in two more coords z_1^0, z_2^0 (resp. w_1^0, w_2^0)
  View z_j^k (resp. w_j^k) as coordinates of points in $V_{\mathbb{C}} = \mathbb{C}^{2n+2}$.
- $\left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \neq 0 \right\}$ group under multiplication $\Rightarrow$
- Closure of $\mathbb{R}$-Minkowski space in $Grass(2, V)$ is

$$\left\{ \begin{pmatrix} z_1^0 & z_2^0 & \cdots & z_1^n & z_2^n \\ -\bar{z}_2^0 & \bar{z}_1^0 & \cdots & -\bar{z}_2^n & \bar{z}_1^n \end{pmatrix} \neq 0 \right\} \subset Grass(2, V)$$

Back to σ

- Define $\sigma \subset V \times V - R$ by scaling

$$\sigma := \left\{ \begin{pmatrix} z_1^0 & z_2^0 & \cdots & z_1^n & z_2^n \\ -e^{i\theta}\bar{z}_2^0 & e^{i\theta}\bar{z}_1^0 & \cdots & -e^{i\theta}\bar{z}_2^n & e^{i\theta}\bar{z}_1^n \end{pmatrix} \;\middle|\; \sum_{j,k} |z_j^k|^2 = 1, \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \quad (4)$$

- We have seen Q pos. linear comb. of propagators $\Rightarrow \sigma \cap R = \emptyset$.
- σ fibres over the closure of $\mathbb{R}$-Minkowski sp. with fibre U_2. (Note U_2 acts on σ by left multiplication.)

$[\sigma] \neq 0 \in H_{4n+4}(V \times V - R, \mathbb{Q})$.

- Show $\int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}v}{Q^{2n+2}} \neq 0$.
- $Q = \sum b_k v_1^k \wedge v_2^k \in \bigwedge^2 V^\vee$; here v_j^k dual to z_j^k, $b_k > 0$.
- $Q(\text{matrix in } (4)) = e^{i\theta} \sum_k b_k(|z_1^k|^2 + |z_2^k|^2)$
- $$d^{2n+2}z \wedge d^{2n+2}w|_\sigma = ie^{(2n+2)i\theta} d\theta \wedge (\text{const.} \neq 0) \cdot (\text{Vol. form of } 4n+3 \text{ sphere})$$
- Factor $e^{i\theta}$ cancels in $\frac{d^{2n+2}z \wedge d^{2n+2}v}{Q^{2n+2}}|_\sigma$.
- $$\frac{d^{2n+2}z \wedge d^{2n+2}v}{Q^{2n+2}}|_\sigma = (\text{const.} \neq 0) d\theta \cdot (\text{Vol. form of } 4n+3 \text{ sphere})$$

Fubini

- $\dim Grass(2, V) = 4n$; $\Omega^{4n}_{Grass} = \mathcal{O}_{Grass}(-2n-2)$.
  Here $Grass(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$; $Q \in \bigwedge^2 V^\vee \cong \Gamma(Grass, \mathcal{O}(1))$.
- Fix $0 \neq \Omega(Grass) \in \Gamma(Grass, \Omega^{4n}_{Grass}(2n+2)) \cong \mathbb{C}$
- $\Omega(Grass)/Q^{2n+2} \in \Gamma(G^0, \Omega^{4n})$
- $$\Omega^{4n+4}_{V\times V-R} \cong \rho^*\Omega^{4n}_{G^0} \otimes \Omega^4_\rho$$
- Integral along fibres of ρ is constant

$$\frac{d^{2n+2}z \wedge d^{2n+2}v}{Q^{2n+2}} \Big/ \frac{\rho^*\Omega(Grass)}{Q^{2n+2}}$$

Q's cancel. Get non-vanishing section of Ω^4_ρ over $V \times V - S$.
Constant multiple of volume form on fibre.

Conclusion

- Γ n loops, $2n+2$ edges. Euclidean propagators p_e, $e \in \text{Edge}(\Gamma)$. Amplitude

$$\mathcal{A}(\Gamma, q_{extern}, m) = \int_{\mathbb{R}^{4n}} \frac{d^{4n}x}{\prod_e p_e}.$$

- $\delta = \{(a_1, \ldots, a_{2n+2}) \mid a_i \geq 0\} \subset \mathbb{P}^{2n+1}(\mathbb{R})$.
- $O = \mathbb{C}^2$, $I = \mathbb{C}^{2n}$, $V = O \oplus I$. $\exists Q_e \in \bigwedge^2 V^\vee$ (interpret as $(2n+2) \times (2n+2)$ matrix) such that

$$\mathcal{A}(\Gamma, q_{extern}, m) = (const.) \int_\delta \frac{\Omega_{2n+1}}{Pfaffian(\sum a_e Q_e)^2}.$$

Questions

- $X : \mathbb{P}^{2n+1}$ hypersurface; $X : Pfaffian(\sum a_e Q_e) = 0$.
- Generalization of graph hypersurface?

  X depends on parameters: ext. momenta, masses.

  Thresholds and monodromy (Cutkosky rules).
- Combinatorics of pfaffians?

  $H_1(\Gamma, \mathbb{Q}) \subset \mathbb{Q}^{\text{edges}}$. $e \rightsquigarrow e^{\vee,2}$ rank 1 quadric

  Pfaffian invariants of configurations?
- Study simpler case of first Symanzik polynomial :(

Riemann-Kempf Theorem (?!)

- C complete, smooth alg. curve; $J = Jac(C)$ Jacobian variety.

  $\rho : C \hookrightarrow J,\ \rho(c) = (c) - (c_0).\ C^{(i)} = Sym^i C.$

  $\rho^i : C^{(i)} \longrightarrow J,\ (c_1, \ldots, c_i) \mapsto \sum(c_j) - i(c_0).$

  $W^i = \rho^{(i)}(C^{(i)}) \subset J.$

  Θ-divisor $W^{g-1} \subset J,\ g = \text{genus } C.$
- $D = c_1 + \cdots + c_i,\ \mathcal{L}(D) = \{f \in k(C) \mid (f) + D \geq 0\}.$

  $\rho^{-1}\rho(c_1, \ldots, c_i) = \mathbb{P}(\mathcal{L}(D)) \subset C^{(i)}.$

Theorem (Riemann)

$D = c_1 + \cdots + c_{g-1}$. *Multiplicity of* Θ *at* $\rho(c_1, \ldots, c_{g-1})$ *equals* $\dim \mathcal{L}(D)$.

Riemann-Kempf Theorem (Kempf's contribution)

$D = c_1 + \cdots + c_i$,

$$\Gamma(C, \mathcal{O}(D)) \otimes \Gamma(C, \Omega^1_C(-D)) \xrightarrow{\psi \otimes \phi \mapsto \psi \cdot \phi} \Gamma(C, \Omega^1_C)$$

$M(D) :=$ rectangular matrix $(\psi_r \cdot \phi_s)$
Tan. sp. to J at $\rho(c_1, \ldots, c_i)$ is the dual $\Gamma(C, \Omega^1_C)^\vee$.

Theorem (Kempf)

The tangent cone to W^i at $\rho(c_1, \ldots, c_i)$ is cut out by the maximal minors of $M(D)$.

Application to First Symanzik Polynomial

Recall the fist Symanzik polynomial of a graph Γ is defined by $\det(\sum_e A_e M_e)$
M_e rk. 1 symmetric associated to $e^\vee : H_1(\Gamma, \mathbb{Q}) \to \mathbb{Q}$.

Theorem (Patterson)

The multiplicity of $S_1(\Gamma)$ at a point $(\ldots, a_e, \ldots)$ equals the corank of the symmetric matrix $\sum a_e M_e$.

Remark

It is easy to give examples of families of symmetric matrices for which this is false. (The stupidest is just a family of 1×1 matrices $\{f(t)\}$ where f has a zero of order > 1 at $t = t_0$.)

Remark

It would be very interesting to have some analog of Kempf's theorem for $S_1(\Gamma)$.

Discussion

- linear information about S_2? Is there an analog of Riemann-Patterson theorem for Pfaffian($\sum a_e Q_e$)?.
- $\pi : \Lambda \to X : S_1(\Gamma) = 0$. $\pi^{-1}(\ldots, a_e, \ldots) = \mathbb{P}(\ker(\sum a_e M_e))$.

Theorem

The motive of Λ is mixed Tate (i.e. "uninteresting").

- Stratify graph polynomial X according to dim. of fibres. Closed strata are analogs of $W^i \subset J$ in curve case.
- Theorem says that there is a subtle link between the geometry of the stratification and the motive of X.

Another Example of Linear Information

- $X \hookrightarrow \mathbb{P}^n$ smooth projective variety.
- $\mathbb{P}^{n,\vee} = \{H \subset \mathbb{P}^n \text{ hyperplane}\}$ dual projective space.
- $X^\vee := \{H \in \mathbb{P}^{n,\vee} \mid X \cap H \text{ singular}\}$. Dual variety. (Usually a hypersurface.)
- Think of $X^\vee$ as analogous to graph hypersurface.
- Analog of $\pi : \Lambda \to X(\Gamma)$?
  $N^\vee \to X$ conormal bundle of $X \subset \mathbb{P}^n$.
  $\mathbb{P}(N^\vee) \subset \mathbb{P}(T_{\mathbb{P}^n}|X) \subset \mathbb{P}(T_{\mathbb{P}^n}) =$ incidence. cor. $\twoheadrightarrow \mathbb{P}^{n,\vee}$
- $X \leftarrow \mathbb{P}(N^\vee) \xrightarrow{\pi} X^\vee \hookrightarrow \mathbb{P}^{n,\vee}$
- $\pi^{-1}(H) = (H \cap X)_{sing}$.

Proof of Patterson's Theorem; Some lemmas

$H \subset \mathbb{Q}^E$ arrangement. $e \in E$, $e^{\vee,2} : H \to \mathbb{Q}$ rank 1 quadric.
$P(H \subset \mathbb{Q}^E) := \det(\sum_e A_e M_e)$; M_e matrix of $e^{\vee,2}$.

Lemma

(i) $\deg_{A_e} P \leq 1$.
(ii) $\frac{\partial}{\partial A_e} P(H \subset \mathbb{Q}^E) = P(H \cap \{e^\vee = 0\} \subset \mathbb{Q}^{E-\{e\}})$.

Lemmas (cont.)

Lemma

Let $P = P(H \subset \mathbb{Q}^E)$ be the arrangement polynomial as above, and let $a = (\dots, a_e, \dots)$ be a point. The multiplicity of P at a is less than or equal to the corank of the matrix $\sum_e a_e M_e$.

Proof.

If the corank is zero, then $\det(\sum a_e M_e) \neq 0$ so the multiplicity is zero. In general, the corank is the dimension of the null space of the quadratic form associated to the symmetric matrix $\sum a_e M_e$. Intersecting with a hyperplane $e^\vee = 0$ can drop that null space dimension by at most 1. It follows that $\frac{\partial^p}{\partial A_{e_1} \cdots \partial A_{e_p}} P(a) = 0$ for $p < \operatorname{corank}(\sum a_e M_e)$. □

Main Lemma

Lemma

Let H be a vec. sp. of dim r over field k, char. k not 2. Let $e_i^\vee \in H^\vee$ span $H^\vee$. Let $Q : H \to k$ be a quadratic form. Let $s = \dim N$ where $N \subset H$ is the null space of Q. Then there exist $i_1, \ldots, i_s$ such that writing $L = \cap_{j=1}^s \ker(e_{i_j}^\vee)$, we have $Q|L$ nondegenerate.

Remark

This lemma together with (ii) from the first lemma implies

$$\frac{\partial^s}{\partial A_{e_1} \cdots \partial A_{e_s}} P(a) \neq 0$$

so the multiplicity at a is exactly equal to the corank, proving the theorem.