Math 205 Problem
April 13, 2009

Exercise 1(ii) on the HW involved a fairly serious integral. Here is the answer (and no, problems on exams will not be so difficult!)
Challenge. Compute

$$\iint_{0\le x,y\le 1}(x^2+y^2)^{1/2}dxdy. \tag{0.1}$$

Use Fubini to rerwite the integration

$$\int_0^1 y\Big(\int_0^1 (1+(x/y)^2)^{1/2}dx\Big)dy \tag{0.2}$$

Now make the change of variable $u = x/y,\ x = uy,\ dx = ydu$. The integral becomes

$$\int_0^1 y^2\Big(\int_0^{1/y}(1+u^2)^{1/2}du\Big)dy \tag{0.3}$$

Integration by parts yields

(0.4)
$$\int(1+u^2)^{1/2}du = u(1+u^2)^{1/2} - \int\frac{u^2du}{(1+u^2)^{1/2}} = u(1+u^2)^{1/2} - \int(1+u^2)^{1/2}du + \int\frac{du}{(1+u^2)^{1/2}}.$$

It follows that

$$\int(1+u^2)^{1/2}du = 1/2\Big(u(1+u^2)^{1/2} + \int\frac{du}{(1+u^2)^{1/2}}\Big). \tag{0.5}$$

On the other hand,

$$\int\frac{du}{(1+u^2)^{1/2}} = \log(u+(u^2+1)^{1/2}) \tag{0.6}$$

(How to spot this? Well, it is a bit of an art. One starts with the formula $d/du(\log(f(u))) = (df/du)/f$ and then one looks for an f... Alternatively, there are lots of tables of elementary integrals on the web.)

Anyway, where are we? Combining (0.5) and (0.6) yields

$$\int(1+u^2)^{1/2}du = 1/2\Big(u(1+u^2)^{1/2} + \log(u+(u^2+1)^{1/2})\Big) \tag{0.7}$$

Now substitute into (0.3)

$$\iint_{0\le x,y\le 1}(x^2+y^2)^{1/2}dxdy = \tag{0.8}$$

$$1/2\int_0^1 y^2\Big(y^{-1}(1+y^{-2})^{1/2}+\log(y^{-1}+(y^{-2}+1)^{1/2})\Big) =$$

$$1/2\int_0^1 \Big((y^2+1)^{1/2}-y^2\log y+y^2\log(1+(y^2+1)^{1/2})\Big)dy.$$

This integral can also be computed as an indefinite integral involving the inverse hyperbolic sin, arcsinh. It is a bit painful to do by hand, but Mathematica yields

$$1/2\int \Big((y^2+1)^{1/2}-y^2\log y+y^2\log(1+(y^2+1)^{1/2})\Big)dy = \tag{0.9}$$

$$y\sqrt{1+y^2}/3+\text{arcsinh}(y)/6-y^3\log(y)/6+y^3\log(1+\sqrt{1+y^2})/6.$$

For a numerical answer, you have but to evaluate these quantities at 0 and 1.