(1) State the Class Equation, and use it to show that $p$-groups have nontrivial centers.

(2) Explain cycle decomposition for permutations $g \in S_n$. Show that conjugacy classes $C_x = \{gxg^{-1} \mid g \in S_n\}$ are in 1–1 correspondence with partitions of $n$, i.e. with equations $n = m_1 + m_2 + \cdots + m_r$ such that the $m_i$ are integers and $1 \leq m_1 \leq \ldots \leq m_r$.

(3) State Sylow’s theorems. Show there are $(p - 1)!$ elements of order $p$ in $S_p$. Show there are $(p - 2)!$ $p$-Sylow subgroups.

(4) State the structure theorem for finitely generated abelian groups. Discuss elementary divisors. Show that the number of distinct (i.e. non-isomorphic) abelian groups of order $p^n$ is equal to the number of partitions of $n$. (For the def. of partitions, see problem 2 above.)

(5) Draw the picture with the 7 points and the 7 lines associated with the simple group $G$ of order 168. (Warning: this picture has, in various bad movies, been used to invoke the devil. Be very careful...) Show that triples of points $p_1, p_2, p_3$ which do not lie on a line in the picture correspond to bases of the vector space $F_2^3$. Show there are 168 such triples. If we ignore the order, i.e. we just consider the set $\{p_1, p_2, p_3\}$, so for example $\{p_1, p_2, p_3\} = \{p_2, p_1, p_3\}$, there are 28 such sets. Prove that $G$ acts transitively on this collection of 28 elements, and the stabilizer of a set is isomorphic to $S_3$. Discuss the structure of the set of 3-Sylow subgroups in $G$. What is $n_3(G)$? (Warning: there is an error in the book on this point.)