MATH270 - May 6 Lecture

Theorem 0.1. *(Maximum Modulus Principle) Let f be a nonconstant analytic function on a connected open set U. Then $|f|$ cannot attains maximum in U, i.e. there exists no $a \in U$ such that $|f(a)| \geq |f(z)|$ for all $z \in U$.*

Proof. Suppose otherwise such a exists. We will show that f is a constant.

Let $M = |f(a)|$. Pick any $b \in U$ and a path $\gamma \subset U$ so that $\gamma(0) = a, \gamma(1) = b$. Since $\gamma([0,1])$ is compact and disjoint from $\mathbb{C}\backslash U$ we have

$$d := \text{distance}(\gamma([0,1]), \mathbb{C}\backslash U) > 0.$$

Applying CIF, we have

$$f(a) \quad = \quad \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{z-a} dz$$

for any $r < d$. Hence,

$$\begin{aligned} M = |f(a)| \quad &\leq \quad \frac{1}{2\pi} \sup_{|z-a|=r} \left| \frac{f(z)}{z-a} \right| \text{arclength}(|z-a| = r) \\ &= \quad \frac{1}{2\pi}\left(\sup_{|z-a|=r} |f(z)|/r \right) 2\pi r \\ &= \quad \sup_{|z-a|=r} |f(z)| \\ &\leq \quad M. \end{aligned}$$

This shows that we must have $|f(z)| = M$ for any z on the circle $|z - a| = r$. This holds for any $r < d$, hence $|f(z)| = M$ for any $z \in D_d(a)$.

Pick a point a_1 on γ such that $|a - a_1| > d/2$ and apply the above argument with a_1 in place of a, we conclude that $|f| = M$ on $D_d(a_1)$. Then pick a point a_2 on γ such that $|a_1 - a_2| > d/2$ and so on. After finitely many step, we get a point a_n such that $b \in D_d(a_n)$. Hence $|f(b)| = M$.

This holds for any $b \in U$, so $|f|$ is constant.

If $M = 0$, we have $f \equiv 0$.

If $M \neq 0$ then $\overline{f} = M^2/f$ is holomorphic. A simple application of CR equations then shows that f must be a constant. □

Corollary 0.2. *Let f be a function holomorphic on a bounded connected set U and continuous on $\overline{U}$. Then*

$$\sup_{z\in U} |f(z)| \leq \sup_{z\in\partial U} |f(z)|$$

where $\partial U = \overline{U}\backslash U$.

Proof. Since $\overline{U}$ is compact, $|f|$ attains maximum in $\overline{U}$. If the maximum is attained in ∂U, we are done. If not, it is attained in U and by previous theorem, f is a constant, hence $\sup_{z\in U} |f(z)| = \sup_{z\in\partial U} |f(z)|$. □

Theorem 0.3. *(Open mapping theorem) Let f be a nonconstant analytic function on a connected open set U. Then $f(U)$ is an open set.*

Proof. We need to show that for any $a \in U$ there is $r > 0$ such that

$$D_r(f(a)) \subset f(U). \quad (*)$$

Let $g(z) = f(z) - f(a)$. Since f is not a constant, neither is g. Hence by a problem in the Midterm, there exists $d > 0$ such that on $D_{d+\epsilon}(a)$, g only vanishes at a. Let

$$2r = \inf_{|z-a|=d} |g(z)| \quad (**).$$

We will show that (*) holds. Suppose not, then there is $w \in D_r(f(a)) \backslash f(U)$. Put

$$h(z) = \frac{1}{f(z) - w}.$$

We have

$$|h(a)| = 1/|f(a) - w| > 1/r.$$

Any for any z such that $|z - a| = d$,

$$|f(z) - w| \geq |f(z) - f(a)| - |f(a) - w| > 2r - r = r,$$

by (**). Hence,$|h(z)| < 1/r$ on $|z - a| = d$.

This contradicts the previous corollary applying to $U = D_d(a)$ and h. □

1. Conformal maps

Definition 1.1. Let f be an analytic map $U \mapsto \mathbb{C}$. We say f is conformal if $f'(z) \neq 0$ for all $z \in U$.

Suppose we have two paths $\gamma_1, \gamma_2 : [-1, 1] \mapsto U$ such that $\gamma_1(0) = \gamma_2(0) = a$. Let $\sigma_1 = f(\gamma_1)$, $\sigma_2 = f(\gamma_2)$. Then

$$\begin{aligned} \sigma_1'(0) &= f'(a)\gamma_1'(0) \\ \sigma_2'(0) &= f'(a)\gamma_2'(0). \end{aligned}$$

So the angle between tangent vectors of σ_1, σ_2 at $t = 0$ is the same as the angle between tangent vectors of γ_1, γ_2 at $t = 0$. And the proportion of the magnitudes of these tangent vectors is also preserved.

An important class of conformal maps is linear fractional transformations (LFTs):

$$f(z) = \frac{az + b}{cz + d}, \text{ where } a, b, c, d \in \mathbb{C} \text{ such that } ad \neq bc.$$

f is injective and maps $\mathbb{C}\backslash\{-\frac{d}{c}\}$ onto $\mathbb{C}\backslash\{\frac{a}{c}\}$. It is conformal since

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

for all z.

We can also view $f : \mathbb{C}^* \to \mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{C} \cup \infty$ and

$$\begin{aligned} f(\infty) &= \frac{a}{c} \\ f(-\frac{d}{c}) &= \infty. \end{aligned}$$

Proposition 1.2. *Let f, g be LFTs. Then*

a/ $f \circ g$ is also a LFT.

b/ The inverse of f is a LFT.

c/ If $f(z_j) = g(z_j)$, $j = 1, 2, 3$ for three distinct points z_1, z_2, z_3 then $f = g$.

Proof. a/ This is a straightforward computation.

b/ The inverse of $f(z) = \frac{az+b}{cz+d}$ is

$$h(w) = \frac{dw - b}{-cw + a}.$$

c/ If $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{a'z+b'}{c'z+d'}$ then we have

$$(az_j + b)(c'z_j + d') - (cz_j + d)(a'z_j + b') = 0,$$

for $j = 1, 2, 3$.

Since

$$(az + b)(c'z + d') - (cz + d)(a'z + b')$$

is a polynomial of degree less than or equal two having three distinct zeroes, it is identically zero.

We conclude that $f(z) = g(z)$ for all z. ∎