Notes for Math 270, Week of May 10-14, 2004

Lectures focused on three ideas:
(1) Holomorphic maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
(2) Conformal maps; the Riemann mapping theorem.
(3) Möbius (or fractional linear) transformations $f(z) = \frac{az+b}{cz+d}$.

By definition $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. as a set, $\widehat{\mathbb{C}}$ is the complex numbers together with an extra point which is called ∞. We give $\widehat{\mathbb{C}}$ the structure of a topological space by describing neighborhoods of ∞. For $R > 0$ we consider $D_R(\infty) := \{\infty\} \cup \{z \mid |z| > 1/R\}$.

Definition 1. *A function $f : D_R(\infty) \to \mathbb{C}$ is analytic at ∞ if the function $g(z) = f(\frac{1}{z})$ is analytic at $z = 0$.*

Intuitively, the mapping $z \mapsto \frac{1}{z}$ identifies $D_R(0) \cong D_R(\infty)$. We use this identification to "carry over" the notion of analyticity which we know on $D_R(0)$ to the corresponding notion on $D_R(\infty)$.

Example 2. *The function $f(z) = \frac{1}{z+1}$ is analytic at ∞, because $g(z) = f(z^{-1}) = \frac{z}{z+1}$ is analytic at $z = 0$. What about $f(z) = e^z$?*

Let $U \subset \mathbb{C}$ be an open set, and let $f : U \to \widehat{\mathbb{C}}$ be a function. Suppose $z_0 \in U$ and $f(z_0) = \infty$.

Definition 3. *f as above is meromorphic at z_0 if $g(z) := \frac{1}{f(z)}$ is analytic at $z = z_0$.*

Strict parallelism would suggest we should call such an f holomorphic at z_0, but it is customary to not use the word holomorphic at a point where the function has a pole.

Example 4. *The function $f(z) = \frac{az+b}{cz+d}$ is meromorphic at the point $z_0 = -d/c$. (We assume $ad - bc \neq 0 \neq c$.) Indeed,*

$$g(z) = 1/f(z) = \frac{cz+d}{az+b}$$

is analytic at $-d/c$ and takes the value 0 there.

As an exercise you should work out what it means for f to be meromorphic when $f : D_R(\infty) \to D_{R'}(\infty)$ with $f(\infty) = \infty$. Then use this to define what it means for $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to be meromorphic. Finally, show all linear fractional transformations are meromorphic as maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Conformal Maps

Let U, V be open sets in $\mathbb{C}$, and let $f : U \to V$ be an analytic function. Suppose there exists a $g : V \to U$ such that $g(f(z)) = z$. The chain rule then tells us

$$1 = \frac{d}{dz}(z) = g'(f(z)) \cdot f'(z).$$

It follows that $f'(z)$ is never zero for $z \in u$.

Definition 5. *An analytic function f on U such that $f'(z)$ is never 0 is said to be conformal on U. U and V are said to be conformally equivalent if there exist $1-1$ onto maps f and g as above.*

The main result on conformal equivalence is:

Theorem 6 (Riemann Mapping Thm)**.** *Let U, V be open sets in $\mathbb{C}$. We assume these sets are contractible, i.e. there should exist continuous mappings*

$$h : U \times [0,1] \to U; \quad k : V \times [0,1] \to V$$

such that $h(u,0) = u, h(u,1) = u_0, k(v,0) = v, k(v,1) = v_0$ where $u_0 \in U$ and $v_0 \in V$ are fixed points independent of u and v. (Intuitively, this means that U and V have no "holes".) We further assume that U and V are distinct from $\mathbb{C}$, i.e. $\mathbb{C} - U \neq \emptyset \neq \mathbb{C} - V$. Then there exist conformal maps $f : U \to V$ and $g : V \to U$ which are $1-1$ onto with $f \circ g = id_V$ and $g \circ f = id_U$. If we fix base points $u_0 \in U$ and $v_0 \in V$ and we require $f(u_0) = v_0$ and $f'(u_0) > 0$, then f and g are unique.

Linear Fractional Transformations

I assume everyone has seen the general linear group in dimension 2

$$GL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0,\ a,b,c,d \in \mathbb{C} \right\}.$$

If we identify $\mathbb{C}^2$ with the column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \in \mathbb{C}$ then we can identify $GL_2(\mathbb{C})$ with the group of linear transformations of $\mathbb{C}^2$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

The composition law for these linear transformations is multiplication of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

The inverse is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

We can think of elements of $GL_2(\mathbb{C})$ as corresponding to linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow f(z) = \frac{az+b}{cz+d}$$

You should check that the product of matrices corresponds to the composition of linear fractional transformations

$$(1) \quad \begin{pmatrix} aa' + bc' & ab' + d'b \\ ca' + c'd & cb' + d'd \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \rightsquigarrow f(g(z)) =$$

$$\frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{(aa' + c'b)z + (ab' + d'b)}{(ca' + c'd)z + (cb' + d'd)}.$$

The derivative of a linear fractional transformation is given by

$$\left(\frac{az+b}{cz+d}\right)' = \frac{ad-bc}{(cz+d)^2}.$$

It vanishes only at $z = -d/c$. You should check that a linear transformation gives rise to a mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is conformal (at least away from ∞. Note we have not discussed the derivative of a mapping at infinity) and has an inverse.

Linear fractional transformations carry "circles" to "circles". By definition a "circle" is defined by an equation

$$Ax + By + C(x^2 + y^2) = D; \; z = x + iy, \; A, B, C < D \in \mathbb{R}$$

If $C = 0$, our "circle" is a line. To show that linear fractional transformations carry circles to circles, we showed that any linear fractional transformation could be written as a composition of three types:

$$z \mapsto az; \quad z \mapsto z + b; \quad z \mapsto 1/z$$

(Here $a, b \in \mathbb{C}$.) Writing $a = \alpha + i\beta$, we get

$$az = (\alpha x - \beta y) + i(\alpha y + \beta x).$$

We want to solve for A', B', C', D' the equation

$$(2) \quad 0 = Ax + By + C(x^2 + y^2) - D =$$

$$A'(\alpha x - \beta y) + B'(\alpha y + \beta x) + C'((\alpha x - \beta y)^2 + (\alpha y + \beta x)^2) - D'.$$

I leave you to work this out. The crucial point is that the cross terms $\alpha\beta xy$ cancel.

The lft $z \mapsto z + b$ carries circles to circles and lines to lines. This is evident from the geometric description of addition for complex numbers.

Finally we need to examine $z \mapsto 1/z$. We have $\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$. Notice

$$(3) \quad A'\frac{x}{x^2+y^2} + B'\frac{-y}{x^2+y^2} + C'(\Big(\frac{x}{x^2+y^2}\Big)^2 + \Big(\frac{y}{x^2+y^2}\Big)^2) - D' = (x^2+y^2)^{-1}\Big(A'x - B'y + C' - D'(x^2+y^2)\Big)$$

This shows that inversion $z \mapsto 1/z$ carries "circles" to "circles".