(1) Define the complex numbers and discuss their properties. You may assume the real numbers to be known. You should include definitions of the real and imaginary parts, the Argand diagram (complex plane), the modulus (absolute value) and the argument of a complex number. You should explain addition and multiplication of complex numbers. Your discussion of these operations should include both a geometric and an algebraic description.

(2) (a) Let \( f : \mathbb{C} \to \mathbb{C} \) be a function. Define what is meant by the statement “\( f \) is analytic at \( z_0 \in \mathbb{C} \)”. Illustrate with examples of functions which are and are not analytic.

(b) State the Cauchy Riemann equations for \( f \).

(3) (a) Use power series to define the functions \( e^z, \cos(z), \sin(z) \). Compute the derivatives of these functions, justifying your arguments with power series.

(b) State the relation between \( e^{iz}, \sin(z), \cos(z) \) and prove this relation using power series.

(c) Define \( \log(z) \) and discuss its well-definedness.

(4) (a) State the chain rule for computing \( \frac{d}{dz}(f(g(z))) \) when \( f \) and \( g \) are analytic functions. (You do not need to prove the chain rule.)

(b) State the chain rule for computing \( \frac{d}{dt}(f(\gamma(t))) \) where \( t \) is a real variable and \( \gamma : \mathbb{R} \to \mathbb{C} \) is a path. (You do not need to prove the chain rule.)

(c) Recall \( f \) is conformal at \( z_0 \) if \( f'(z_0) \) exists and is non-zero. Using (b), show that conformal maps preserve angles. Draw the picture to illustrate your discussion.

(5) (a) Define the path integral (or contour integral) \( \int_\gamma f(z)dz \). You may assume familiarity with path integrals in \( \mathbb{R}^2 \).

(b) Assuming \( f \) analytic, use the chain rule to prove for \( \gamma : [0,1] \to \mathbb{C} \) a path

\[
\int_\gamma f'(z)dz = f(\gamma(1)) - f(\gamma(0)).
\]

(6) Compute the following integrals.
(a) \[
\int_{1+i}^{2+i} zdz
\]
(b) \[
\int_{z_0}^{z_1} e^z dz \quad \text{(Integrate the power series for } e^z \text{ term by term)}
\]
(c) \[
\int_S \text{arg}(z) dz
\]
where \( S \) is the unit circle \( \{|z| = 1\} \).
(d) Use polar coordinates \( z = |z| e^{i \text{arg}(z)} \) to compute
\[
\int_{\{|z|=1\}} \frac{dz}{z}
\]

(7) Cauchy’s theorem says that for \( f \) analytic on and inside a simple closed curve \( \gamma \), one has
\[
\int_\gamma f(z) dz = 0.
\]
(a) State Cauchy’s integral formula.
(b) Discuss the proof of Cauchy’s integral formula using Cauchy’s theorem. You need not be too precise, but draw the picture to illustrate your logic.
(c) Use Cauchy’s integral formula to compute
\[
\int_{\{|z|=2\}} \frac{dz}{z(z-1)}; \quad \int_{\{|z|=2\}} \frac{zdz}{z(z-1)}
\]
Hint: First compute
\[
\int_{\{|z|=2\}} \frac{dz}{z} \quad \text{and} \quad \int_{\{|z|=2\}} \frac{dz}{z-1}.
\]