Motivic Γ-functions

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“Brothers, we are treading, where the saints have trod”

- “We”:= Joint work with M. Vlasenko and F. Brown
- “saints”:= Iterated integrals: Chen, Hain, Beilinson
- “more saints”:= Mixed Tate Hodge Structures, MZV’s: Deligne, Ihara, Goncharov, F. Brown
- “still more saints”:= Motivic Γ-functions: V. Golyshev, D. Zagier
- “and yet more saints”:= Relative completion: Deligne, Hain
- A shout-out to the physicists for first understanding the importance of these *periods*

Iterated Integrals, $Li_n(z)$

- Iterated integral (example)

$$Li_2(z) := -\int_0^z \log(1-t)\frac{dt}{t} = \int_0^z \Big(\int_0^t \frac{du}{1-u}\Big)\frac{dt}{t} =: \int_0^z \frac{dt}{t}\circ\frac{dt}{1-t}$$

- General iterated integral formula. $w_i = w_i(t)dt$ $\int_a^b w_1 \circ \cdots \circ w_r dt := \int_a^b w_1(t)\Big(\int_a^t w_2\circ\cdots\circ w_r\Big)dt.$
- Exercise: $\zeta(n) = \int_0^1 \underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n-1}\circ\frac{dt}{1-t}$

Multiple Polylogarithms and Multiple Zeta Values

- $s := (s_1, \dots, s_\ell)$
- $Li_s(t_1, \dots, t_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{t_1^{n_1} \cdots t_\ell^{n_\ell}}{n_1^{s_1} \cdots n_\ell^{s_\ell}}$
- $\zeta(s) := Li_s(1, \dots, 1) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}}$
- Integral formula for $\zeta(s)$. Define

$$\eta_{s_i} = \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{s_i - 1} \circ \frac{dt}{1-t}; \quad \text{Then } \zeta(s) = \int_0^1 \eta_{s_1} \circ \cdots \circ \eta_{s_\ell}.$$

MZV's (Cont.)

- Example:

$$\zeta(3,2) = \sum_{m>n\geq 1} \frac{1}{m^3 n^2} = \int_0^1 \frac{dt}{t} \circ \frac{dt}{t} \circ \frac{dt}{1-t} \circ \frac{dt}{t} \circ \frac{dt}{1-t}.$$

- Remark: For the iterated integral of a MZV to converge, we must have $\frac{dt}{t}$ on the left and $\frac{dt}{1-t}$ on the right.

MZV's (Tangential basepoints)

- MZV's are periods associated to the unipotent fundamental group of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. To develop this theory, one needs tangential basepoints. We will not pursue this. See [], [], and the references cited there.

Iterated Integrals (elementary topological theory)

- X punctured Riemann surface, $\omega_1, \ldots, \omega_n$ holomorphic 1-forms on X, $\sigma : [0,1] \to X$ smooth path, $\Delta = \{0 \leq t_1 \leq \cdots \leq t_n \leq 1\} \subset [0,1]^n$.

$$() \quad \int_\sigma \omega_1 \circ \cdots \circ \omega_n := \int_{\sigma^n(\Delta)} \omega_1 \wedge \cdots \wedge \omega_n.$$

  Here $\sigma^n : \Delta \subset [0,1]^n \to X^n$, and $\omega_1 \wedge \cdots \wedge \omega_n$ is the evident holomorphic n-form on X^n.
- Suppose $a = \sigma(0),\ b = \sigma(1)$. Define $X_i^n = \{(x_1, \ldots, x_n) \mid x_i = x_{i+1}\}$ large diagonals. Define

$$D^n_{a,b} := \{a\} \times X^{n-1} \cup X^{n-1} \times \{b\} \cup \bigcup_{i=1}^{n-1} X_i^n$$

- $\sigma^n(\Delta) \in H_n(X^n, D^n_{a,b})$; $\omega_1 \wedge \cdots \wedge \omega_n \in H^n(X^n, D^n_{a,b})$.
- Iterated integral () is a period $H^n_{DR}(X^n, D^n_{a,b}) \times H_{n,\mathrm{Betti}}(X^n, D^n_{a,b}) \to \mathbb{C}$.

Beilinson-Chen theory

- $X/\mathbb{C}$ algebraic variety, $x \in X(\mathbb{C})$ basepoint, $\pi_1(X, x)$ fundamental group, $I \subset \mathbb{Q}(\pi_1(X, x))$ augmentation ideal in group ring.
- Unipotent local system $:=$ representation of $\mathbb{Q}(\pi_1(X, x))/I^n$ for some n.
- Chen: $\mathbb{C}(\pi_1(X, x))/I^n$ in terms of differential forms on X. [].
- Beilinson: $\mathbb{Q}(\pi_1(X, x))/I^n$ in terms of homology. (Motivic theory.) [].

Chen theory (in a nutshell)

- $\sigma^n(\Delta) \in H_n(X^n, D^n_{a,b})$ easy!!
- $\omega_1 \wedge \cdots \wedge \omega_n \in H^n(X^n, D^n_{a,b})$ not so clear!!

Beilinson theory (in another nutshell)

- Analyse $H^n(X^n, D^n_{a,b})$ using the Leray spectral sequence for $pr_1 : X^n \to X$.
- Key point: fibre $pr_1^{-1}(x) = (X^{n-1}, D^{n-1}_{x,b})$ for $x \neq a$.
- Theorem (Beilinson)

$$H_n(X^n, D^n_{a,b}) \cong \mathbb{Q}(\pi_1(X, \{a, b\}))/I^{n+1}\mathbb{Q}(\pi_1(X, \{a, b\})).$$

  Here $\mathbb{Q}(\pi_1(X, \{a, b\}))$ is the vector space spanned by homotopy classes of paths from a to b. It has an action of $\pi_1(X, a)$ by composition of paths.
- Variant $\widetilde{D}^n_a$ to deal with $\mathbb{Q}(\pi_1(X, a))$. (Details omitted. See [], Prop. 3.4)

$$H_n(X^n, \widetilde{D}^n_a) \cong \mathbb{Q}(\pi_1(X, a))/I^{n+1}.$$

Tannakian Categories

- Category $\mathcal{C}$ of unipotent local systems on X is Tannakian
- Fibre functor
  $\mathcal{C} \to$ Vector spaces, $\quad V \mapsto V_x =$ fibre of V at x
- $\mathcal{C} = \mathrm{Rep}(\mathcal{G}),\ \mathcal{G} = \mathrm{Spec}(A)$, affine progroupscheme.
- Filtration fil_A with $gr_n A \cong H^n(X^n, \widetilde{D}^n_a)$.

$$\text{Ex: } gr_1 A \cong H^1(X, a; \mathbb{Q}) \cong \mathrm{Hom}(I/I^2, \mathbb{Q}).$$

Mixed Tate Hodge Structures

- $H = H_{\mathbb{Q}}$ f.d. $\mathbb{Q}$-vector space with increasing filtration $W_2 H$.
- Decreasing (Hodge) filtration $F^H_{\mathbb{C}}$.
- Two filtrations opposite.

$$H_{\mathbb{C}} = W_{2n} \oplus F^{n+1}; \quad gr^W_{2n} H_{\mathbb{C}} = F^n gr^W_{2n} H_{\mathbb{C}} \cap \overline{F^n gr^W_{2n} H_{\mathbb{C}}}$$

$$H_{\mathbb{C}} = \begin{pmatrix} F^n \\ W_{2n-2} \end{pmatrix}$$

- Ex:
  $\dim_{\mathbb{Q}} \mathbb{Q}(n) = 1, \; \mathbb{Q}(n) = W_{-2n}\mathbb{Q}(n) \supset W_{-2n-2}\mathbb{Q}(n) = (0)$.
- Ex: $\dim_{\mathbb{Q}} H = 2; \quad H = W_0 H \supset W_{-2} H \supset W_{-4} H = (0)$.

$$0 \to W_{-2} H \to H \to gr^W_0 H \to 0; \quad F^0 H_{\mathbb{C}} \cong gr^W_0 H_{\mathbb{C}}$$

Mixed Tate Hodge Structures (Tannakian structure)

- Category $\mathbb{Q}MTHS$ graded tannikian;
  fibre functor $(H_{\mathbb{Q}}, W_* H_{\mathbb{Q}}, F^* H_{\mathbb{C}}) \mapsto gr^W H_{\mathbb{Q}}$.
- $\mathbb{Q}MTHS \cong \text{Rep}(\mathcal{G})$; $\mathcal{G} = \text{Spec}(A)$.
- Describe A (Goncharov-Zhu, [])

Mixed Tate Hodge Structures (Goncharov-Zhu Hopf algebra construction)

- Graded $\mathbb{Q}$-vector space structure on A:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots;$$

$$A_0 = \mathbb{Q},\ A_1 = \mathbb{C}^*_{\mathbb{Q}},\ A_j = \mathbb{C}^*_{\mathbb{Q}} \otimes \mathbb{C}^{\otimes j-1},\ j \geq 2$$

Mixed Tate Hodge Structures (Goncharov-Zhu Hopf algebra comultiplication)

- Comultiplication
  $\Delta : A \to A \otimes_{\mathbb{Q}} A;\ \Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n + \overline{\Delta}(a_n)$
  $\overline{\Delta}(a_n) = \oplus_{\substack{p+q=n \\ p,q\geq 1}} \Delta_{p,q}(a_n)$
  $\Delta_{p,q} : A_{p+q} \to A_p \otimes A_q;$
  $\Delta_{p,q}(a_1 \otimes \cdots \otimes a_{p+q}) =$
  $\Big(a_1 \otimes \cdots \otimes a_p\Big) \otimes \Big(\exp(a_{p+1}) \otimes a_{p+2} \otimes \cdots \otimes a_n\Big)$

Mixed Tate Hodge Structures (Goncharov-Zhu Hopf algebra multiplication)

- Multiplication: $T(\mathbb{C}) := \bigoplus_{n\geq 0} \mathbb{C}^{\otimes_{\mathbb{Q}} n}$
- $pr : T(\mathbb{C}) \to A$, $pr(c_1 \otimes \cdots \otimes c_n) = \exp(c_1) \otimes c_2 \cdots \otimes c_n$.
- m': $T(\mathbb{C}) \otimes_{\mathbb{Q}} T(\mathbb{C}) \to T(\mathbb{C})$:

$$m'(x_1 \otimes \cdots \otimes x_p, y_1 \otimes \cdots \otimes y_q) =$$
$$x_1 \otimes m'(x_2 \otimes \cdots \otimes x_p, y_1 \otimes \cdots \otimes y_q) + y_1 \otimes m'(x_1 \otimes \cdots \otimes x_p, y_2 \otimes \cdots \otimes y_q)$$
$$- \frac{x_1 y_1}{2\pi i} \otimes 2\pi i \otimes m'(x_2 \otimes \cdots \otimes x_p, y_2 \otimes \cdots \otimes y_q)$$

- $\exists m : A \otimes_{\mathbb{Q}} A \to A$ (commutative multiplication) defined so that $pr \circ m' = m \circ (pr \otimes pr)$.

Commutative Multiplication on A. (Example)

- $m : A_1 \otimes A_1 = \mathbb{C}^*_{\mathbb{Q}} \otimes \mathbb{C}^*_{\mathbb{Q}} \to A_2 = \mathbb{C}^*_{\mathbb{Q}} \otimes \mathbb{C}$.

$$m(a \otimes b) := a \otimes \log(b) + b \otimes \log(a) - \exp\left(\frac{\log a \log b}{2\pi i}\right) \otimes 2\pi i.$$

Repns of $\mathcal{G} = \mathrm{Spec}\, A \leftrightarrow A$-comodules (Example)

- $A_0 \oplus A_1 \supset M_x := \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot x, \;\; x \in \mathbb{C}^*$.
- $\nabla : M_x \to A \otimes M_x; \;\; \nabla(1) = 1 \otimes 1, \;\; \nabla(x) = x \otimes 1 + 1 \otimes x$
- $M_x \cong H^1(\mathbb{G}_m, \{1, x\}; \mathbb{Q})$.
- $$\begin{array}{ccccccccc} 0 & \to & H^0(\{1,x\})/H^0(\mathbb{G}_m) & \to & H^1(\mathbb{G}_m, \{1,x\}) & \to & H^1(\mathbb{G}_m) & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \mathbb{Q} & \to & M_x & \to & \mathbb{Q}(-1) & \to & 0 \end{array}$$
- Remark: If $H^*(X)$ is mixed Tate, then so is $\pi_1(X, x)$ and any related HS's. Example: $X = \mathbb{P}^1 - \{0, 1, \infty\}$.

Relative Completion (Deligne, Hain)

- $X/\mathbb{C}$ affine Riemann surface, $\Gamma := \pi_1(X, x)$, R a reductive algebraic group, $\rho : \Gamma \to R(\mathbb{C})$ representation with Zariski dense image.
- $\mathcal{C}$ category of local systems $\mathcal{L}$ on X (or equivalently representations of Γ) together with a filtration $fil_{\mathcal{L}}$ such that $gr_{\mathcal{L}}$ is an R-representation. (I.e. action of Γ on $gr_{\mathcal{L}}$ factors through ρ.)
- $\mathcal{C}$ Tannakian category, Tannaka group $\mathcal{G}$. Diagram ($\mathcal{U}$ unipotent)

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{U} & \to & \mathcal{G} & \to & R & \to & 1 \\ & & & & \uparrow \tilde{\rho} & & \uparrow \rho & & \\ & & & & \Gamma & = & \Gamma & & \end{array}$$

Motives Associated to Relative Completion

- Assume local systems M on X associated to representations of R are motivic; i.e.

$$M = R^n f_{,Betti} \mathbb{Q}_Y, \quad f : Y \to X.$$

- Example: $R = SL_2$, X modular curve.
- Want to understand motives and periods arising from iterated extensions of M's.

Relative Completion; Basic Lemma

Lemma

M a $\mathcal{G}$-representation. $\widetilde{M}$ associated local system (via $\tilde{\rho}$). Then $H^1(\mathcal{G}, M) \cong H^1(X, \widetilde{M})$.

Proof.

Pure thought. □

- $\mathcal{G} = \mathrm{Spec}\,(A)$. Try to understand A as a Hopf algebra and as a motive.

1-Cocycles

- M $\mathcal{G}$-module (f.d. vector space, $\mathcal{G}$-repn),

$$Z^1(\mathcal{G}, M) := \\ \{f : \mathcal{G} \to M \text{ polynomial map} \mid f(g_1 g_2) = g_1 f(g_2) + f(g_1)\}$$

- $Z^1(\mathcal{G}, M)$ is a $\mathcal{G}$-module; $(gf)(h) := g(f(g^{-1}hg))$.
- Exact sequence of $\mathcal{G}$-modules

$$0 \to H^0(\mathcal{G}, M) \to M \xrightarrow{\partial} Z^1(\mathcal{G}, M) \to H^1(\mathcal{G}, M) \to 0;$$
$$\partial(m)(g) := gm - m.$$

- Assume $H^0(\mathcal{G}, M) = (0)$. Then

$$0 \to M \xrightarrow{\partial} Z^1(\mathcal{G}, M) \to H^1(\mathcal{G}, M) \to 0$$

is the universal extension:
$\mathrm{Ext}^1_{\mathcal{G}}(\bullet, M) \cong \mathrm{Hom}_{vect.sp.}(\bullet, H^1(\mathcal{G}, M))$

1-Cocycles (cont.)

- $M_1, \ldots, M_n$ $\mathcal{G}$-representations. Assume $H^0(\mathcal{G}, M_i \otimes M_{i+1} \cdots \otimes M_j) = (0), \forall i \leq j$.
- Construction: $Z_1^1 := Z^1(\mathcal{G}, M_1)$; $Z_i^1 := Z^1(\mathcal{G}, Z_{i-1}^1 \otimes M_i)$, $i \geq 2$.
- Structure: Z_i^1 filtration; graded pieces $M_1 \otimes \cdots \otimes M_i, M_2 \otimes \cdots \otimes M_i \otimes V_2, \ldots, M_i \otimes V_i, V_{i+1}$. V_j vector spaces with trivial $\mathcal{G}$-action.
- $\widetilde{M}_i$ local system associated to M_i, $x \in X$ basepoint, $X_{i,i+1}^n \subset X^n$ large diagonals

$$_xD^n := \{x\} \times X^{n-1} \cup X_{1,2}^n \cup \cdots \cup X_{n-1,n}^n \subset X^n.$$

1-Cocycles (III)

Theorem

Assume the M_i are R-representations. Define the local system $S_n,\ x \mapsto H^n(X^n, {}_xD^n; M_1 \boxtimes \cdots \boxtimes M_n)$. Then S_n admits a filtration with grS_n the local system associated to a representation of R. Under the identification between such filtered local systems on X and representations of $\mathcal{G}$, we have $S_n \leftrightarrow Z^1_n$.

Example

For $n = 1$ we have $0 \to M_{1,x} \to H^1(X, x; M_1) \to H^1(X, M_1) \to 0$. Allowing x to vary yields and exact sequence of local systems $0 \to M_1 \to S_1 \to H^1(X, M_1)_X \to 0$

Matrix Coefficients

- $\theta : \mathcal{G} \to \mathrm{Aut}(V)$ representation. $v \in V,\ w \in V^\vee$.
  Matrix coefficient $g \mapsto \langle gv, w \rangle \in \mathcal{O}_\mathcal{G}$
  $\mathcal{O}_\mathcal{G} \supset \langle V, V^\vee \rangle$ =: Vector space spanned by such
- $\langle V, V^\vee \rangle \subset \mathcal{O}_R \subset \mathcal{O}_\mathcal{G}$ iff $\mathcal{U}$ acts trivially on V.
- $m : \mathcal{O}_\mathcal{G} \to \mathcal{O}_\mathcal{G} \otimes \mathcal{O}_\mathcal{G}$ comultiplication. $\overline{m} = m - pr_1 : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$.
  $\overline{m} : \mathcal{O}_\mathcal{G}^{\otimes n} \to \mathcal{O}_\mathcal{G}^{\otimes n+1}$; $\overline{m}(f_1 \otimes \cdots \otimes f_n) := \overline{m}(f_1) \otimes f_2 \otimes \cdots \otimes f_n$.

Filtration on $\mathcal{O}_\mathcal{G}$

- $fil_*\mathcal{O}_\mathcal{G}: \; fil_{-1} = (0), \; fil_0 = \mathcal{O}_R \subset \mathcal{O}_\mathcal{G}$,

$$fil_n\mathcal{O}_\mathcal{G} := (\overline{m}^*)^{-n}(\mathcal{O}_R \otimes \mathcal{O}_\mathcal{G}^{\otimes n}) \subset \mathcal{O}_\mathcal{G}.$$

- $I_\mathcal{U} \subset k(\mathcal{G})$ ideal gen. by $\{u - 1 \mid u \in \mathcal{U}\}$. Equivalent def. of $fil_*\mathcal{O}_\mathcal{G}$:

$$fil_{-1} = (0); \; fil_0 = \mathcal{O}_R;$$
$$fil_n = \{f \in \mathcal{O}_\mathcal{G} \mid I_\mathcal{U} \cdot f \subset fil_{n-1}\mathcal{O}_\mathcal{G}, \; n \geq 1\}$$

- V representation of $\mathcal{G}$. Then $\langle V, V^\vee \rangle \xrightarrow{\iota} fil_n\mathcal{O}_\mathcal{G} \Leftrightarrow I_\mathcal{U}^{n+1}V = (0)$. ($\iota$ not usually an inclusion)

Example

- Fix $M_1, \dots, M_n$ and Z_n^1 as above.
- Fixing a basepoint $x \in X$, have $Z_n^1 \cong H^n(X^n, {}_xD^n; M_1 \boxtimes \cdots \boxtimes M_n)$. ($Z_n^1$ is motivic).
- Take $V = Z_n^1$ in the above construction; $Z_n^1 \otimes Z_n^{1,\vee} \to fil_n\mathcal{O}_{\mathcal{G}}$
- General fact: Coordinate ring of Tannaka group of a Tannakian category is an ind object in the category.
- Identify (over $\mathbb{C}$) $Z_n^1 = H^n_{DR}(X^n, {}_xD^n; M_1 \boxtimes \cdots \boxtimes M_n)$, $Z_n^{1,\vee} = H_{n,Betti}$.
- Recover (generalized) iterated integrals as composition

$$Z_n^1 \otimes Z_n^{1,\vee} \to \mathcal{O}_{\mathcal{G}} \xrightarrow{\text{eval. at } 1} \mathbb{C}$$

Periods for Connections on Curves

- U smooth open curve/$\mathbb{C}$; M algebraic connection on U

$$\nabla : M \to M \otimes_{\mathcal{O}_U} \Omega^1_U; \quad \nabla(fm) = f\nabla(m) + df \otimes m.$$

$$H^1_{DR}(U, M) := \mathrm{Coker}\Big(\nabla : H^0(U, M) \to H^0(U, M \otimes \Omega^1_U)\Big)$$

- $M^\vee_{an} := M^\vee \otimes \mathcal{O}_{U,an}$, $\mathcal{M}^\vee := \ker(\nabla_{an} : M^\vee_{an} \to M^\vee_{an} \otimes \Omega^1_{U,an})$ $\mathcal{M}^\vee =:$ space of solutions for M
- period pairing:

$$(\bullet, \bullet) : H_1(U, \mathcal{M}^\vee) \times H^1_{DR}(U, M) \to \mathbb{C}$$

Comments and Examples

- We assume ∇ has regular singular points. (Not necessary: exponential motives, rapid decay homology, Bessel functions as periods,...)
- We assume $U \hookrightarrow \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$ so $\Omega^1_U = \mathcal{O}_U \frac{dt}{t}$.
- Basic object of interest: $M = H^n_{DR}(X/U)$ with X/U smooth and projective.
- Example: $X : y^2 = (1-t)^{-1}$, $U = \mathbb{P}^1 - \{0, 1, \infty\}$. $M = \mathcal{O}_U \oplus \mathcal{O}_U y, \ \nabla(y) = \frac{ydt}{2(1-t)}$
- Period calculation: $\sigma : (0,1) \to U$ closed path, $\mu : M \to \mathcal{O}_{U^{an}}$ defined by local analytic section in nghbd of σ.

$$\mu(y) = (1-t)^{-1/2}, \mu(y)|_\sigma \in H_1(U, \mathcal{M}_\vee); \quad ydt/t \in H^1_{DR}(U, M);$$

$$\text{Period} = \int_\sigma (1-t)^{-1/2} dt/t = \int_\sigma \langle \mu, m \rangle \omega; \quad m = y \in M, \ \omega = dt/t$$

Mellin transform

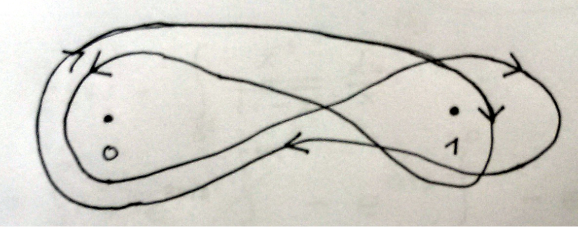
- Connections can be tensored.
- $\nabla_{\mathcal{O},s}$ connection on $\mathcal{O}$, $s \in \mathbb{C}$, $\nabla_{\mathcal{O},s}(1) := sdt/t$
- $\nabla_{M,s} : M \to M \otimes \Omega^1_U,\ \nabla_{M,s}(m) = \nabla_M(m) + smdt/t$
- Period of $\nabla_{M,s}$: $\int_\sigma t^{-s}\langle \mu, m\rangle\omega$.

Example

- Example from previous page:
- $\int_\sigma \frac{t^{-s}}{(1-t)^{1/2}} \frac{dt}{t} = 2(1 - e^{2\pi i s}) \frac{\Gamma(-s)\Gamma(1/2)}{\Gamma(\frac{1}{2}-s)}$

(h)

Figure: σ



Motivic Gamma Functions (Golyshev)

- $\xi := \mu|_\sigma \in H_1(U, \mathcal{M}^\vee)$
  $m \otimes \frac{dt}{t} \in H^1_{DR}(U, M) := (M \otimes \Omega^1_U)/\nabla(M)$.
- *Motivic Gamma function*

$$\Gamma_\xi(s) := \int_\sigma t^{-s} \langle \mu, m \rangle \frac{dt}{t}$$

is the Mellin transform of the period $(\xi, m \otimes \frac{dt}{t})$. (Definition due to Golyshev).

Difference Equations

- $U \subset \mathbb{G}_m,\ D = td/dt,\ D : M \to M$ differential operator
- $m \in M$.
  $L(D)(m) = 0,\ L(D) = p_0(D) + tp_1(D) + \cdots + t^r p_r(D)$;
  p_i polynomials with $\mathbb{C}$-coefficients.
- $\xi = \mu|_\sigma \in H_1(U, \mathcal{M}^\vee)$. Consider motivic Gamma function

$$\Gamma_\xi(s) := \int_\sigma t^{-s} \langle \mu, m \rangle \frac{dt}{t}$$

- $\Gamma_\xi(s)$ satisfies *difference equation*

$$\sum_{j=0}^{r} p_j(-s-j)\Gamma_\xi(s+j) = 0.$$

Taylor expansions of Mellin Transforms

- $\Gamma_\xi(s) = \sum(-s)^n \int_\sigma \frac{(\log t)^n}{n!}\langle \mu, m\rangle \frac{dt}{t}$
- Coefficients are iterated integrals
  $\int \langle \mu, m\rangle \frac{dt}{t} \circ \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \cdots \circ \frac{dt}{t}}_{n}$
- Objective: Interpret these periods geometrically. Relate Taylor coefficients for Mellin transform of a period of M to variation of monodromy. (Inspired by results of Golyshev-Zagier []).
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Frobenius Method-MUM Points

- $M = H^r_{DR}(X/U)$, X/U smooth, projective family.
- G-Z case: $M = H^2_{DR}(X/U)$, X/U smooth family of $K3$-surfaces with Picard rank 19.
- $\mathcal{M} = M_{an}^{\nabla=0}$ local system on U_{an}. $p \in \mathbb{P}^1 - U$.
- Assume monodromy for M around p is unipotent with only one Jordan block.

Frobenius Method-MUM Points (local construction.)

- Δ^* punctured disk with coordinate t. $\mathcal{O}$ ring of analytic functions on Δ^* meromorphic at 0.
- Fix n, r>0 integers ($r = \text{rank}(M)$). Define $\mathbb{E}_n$ free $\mathcal{O}$-module rank $n+1+r$.

$$\mathbb{E}_n := \mathcal{O}e_{-n-1} \oplus \cdots \oplus \mathcal{O}e_0 \oplus \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_{r-1}$$

- Connection on $\mathbb{E}_n$ $\nabla e_i = e_{i-1}\frac{dt}{2\pi it}, \quad \nabla(e_{-n-1}) = 0.$
- Define $\ell(t) := \log(t)/2\pi i$. Horizontal sections of $\mathbb{E}_n$:

$$e_{-n-1},\ \ell e_{-n-1} - e_{-n},\ \frac{\ell^2(t)}{2!}e_{-n-1} - \ell(t)e_{-n} + e_{-n+1}, \ldots$$

- Monodromy $\exp(-N)$, $N(e_i) = e_{i-1},\ N(e_{-n-1}) = 0.$

Frobenius Method-MUM Points (Solutions)

- Dual connection

$$\nabla^\vee e_i^\vee = \frac{-e_{i+1}^\vee dt}{2\pi i t}$$

- Horizontal sections of $\mathbb{E}_n^\vee$ (solutions for $\mathbb{E}_n$)

$$\rho_k(t) := \frac{\ell^k(t)}{k!} e_{r-1}^\vee + \frac{\ell^{k-1}(t)}{(k-1)!} e_{r-2}^\vee + \cdots + \ell(t) e_{r-k}^\vee + e_{r-k-1}^\vee$$

- $N^\vee(e_i^\vee) = e_{i+1}^\vee, \; N^\vee(e_{r-1}^\vee) = 0.$

$$\exp(-\ell(t)N^\vee)\rho_k(t) = e_{r-1-k}^\vee.$$

The Extension

- Define $\mathbb{F}_n = \mathcal{O}e_{-n-1} \oplus \cdots \oplus \mathcal{O}e_{-1} \subset \mathbb{E}_n$, subconnection

$$0 \to \mathbb{F}_n \to \mathbb{E}_n \to \mathbb{E}_n/\mathbb{F}_n \to 0; \quad \mathbb{E}_n/\mathbb{F}_n \cong M_{an}$$

- Note connection on Δ^* determined by its monodromy.
- $\mathbb{F}_n \cong (\mathrm{Sym}^n K_t)(1)$ as variation of Hodge structure. (1) means tensor with $\mathbb{Q}(1)$. (Details omitted.)
- $K_t \leftrightarrow \frac{dt}{t} \in H^1_{DR}(\Delta^*) = \mathrm{Ext}^1_{\mathrm{connection}}(\mathcal{O}, \mathcal{O})$.
- K_t is the Kummer variation of HS associated to the unit $t \in \mathcal{O}^\times$.

The Extension (Hodge Structure)

- Endow $\mathbb{E}_n$ with Hodge filtration to make

$$0 \to (\mathrm{Sym}^n K_t)(1) \to \mathbb{E}_n \to M_{an} \to 0$$

  a sequence of variations of HS.
- M as variation of HS; $0 \subset F^r M \subset \cdots \subset F^0 M = M$.
- $h \in F^r M, \; L(D)(h) = 0, \; M \cong \mathcal{D}/\mathcal{D}L$
  (given by $h \mapsto 1 \in \mathcal{D} := \mathcal{O}(D)$)
- Frobenius method (compare [], section???):

$$\phi^{an}(t) := \phi_0^{an}(t)e_{r-1} + \phi_1^{an}(t)e_{r-2} + \cdots + \phi_{r-1}^{an}(t)e_0 + \cdots + \phi_{r+n}^{an}(t)e_{-n-1} \in \mathbb{E}_n$$

- $D^n L\phi^{an} = 0.$

Frobenius method and solutions

$$\phi_k := \langle \rho_k, \phi^{an} \rangle = \frac{\ell(t)^k}{k!}\phi_0^{an}(t) + \frac{\ell(t)^{k-1}}{(k-1)!}\phi_1^{an}(t) + \cdots + \ell(t)\phi_{k-1}^{an} + \phi_k^{an}$$

- $L(\phi_0) = \cdots = L(\phi_{r-1}) = 0$. (Classical Frobenius method). Also, from [], section 2.1, formula 2.5

$$() \quad L(\phi_{r-1+j}) = \frac{(\log(t))^{j-1}}{(j-1)!}, \ 1 \le j \le n+1.$$

(Frobenius method for inhomogeneous solutions.)

The Extension (Hodge Structure, cont.)

- Image of $\phi^{an}(t)$ is

$$h = \phi_0^{an}(t)e_{r-1} + \phi_1^{an}(t)e_{r-2} + \cdots + \phi_{r-1}^{an}(t)e_0 \in \mathbb{E}_n/\mathbb{F}_n \cong M_{an}.$$

- $s : M_{an} \to \mathbb{E}_n$, $\mathcal{O}$-linear splitting,

$$s(D^p h) = D^p \phi^{an}(t) \quad 0 \leq p \leq r-1$$

- Define $F^i\mathbb{E}_n := s(F^i M_{an}),\ 0 \leq i \leq r;$
$F^i\mathbb{E}_n = F^i((\mathrm{Sym}^n K_t)(1)) + s(M_{an}),\ i < 0.$

Griffiths Transversality

- $\nabla_D : \mathbb{E}_n \to \mathbb{E}_n$. Griffiths transversality says

$$\nabla_D(F^i\mathbb{E}_n) \subset F^{i-1}\mathbb{E}_n.$$

- Show $D^k\phi^{an} \in F^{r-k}\mathbb{E}_n$.
- Case: $k < r$. In this case $D^k\phi^{an} = D^k s(h) = s(D^k h) \in s(F^{r-k}M) = F^{r-k}\mathbb{E}_n$.
- Case $k \geq r$. Write $L = D^r + P$, $\deg_D(P) < r$.

$$D^k = D^{k-r}L - D^{k-r}P$$

- Sufficient by induction on k to show

$$D^{k-r}L\phi^{an} \in F^{r-k-1}\mathbb{E}_n.$$

Griffiths Transversality (cont.)

- Write $L\phi^{an} = B_r(t)e_{-1} + \cdots + B_{r+n}(t)e_{-n-1}$. From () above get

$$\frac{(\log t)^{k-r}}{(k-r)!} = L\phi_k = \langle \rho_k, L\phi^{an} \rangle = \langle \rho_k, B_r e_{-1}^{\vee} + \cdots + B_{r+n} e_{-n-1}^{\vee} \rangle$$

- the $B_j(t)$ are single-valued and do not involve $\log(t)$. From formula for ρ_k it follows that $B_1 = 1, B_j = 0, j > 1$.
- Thus $L\phi^{an} = e_{-1},\ D^j L\phi^{an} = e_{-1-j} \in F^{-1-j}\mathbb{F}_n \subset F^{-1-j}\mathbb{E}_n$.
- Note $F^{-j}\Big((\mathrm{Sym}^n(K_t))(1)\Big) = \mathcal{O}e_{-1} \oplus \cdots \oplus \mathcal{O}e_{-j}$.

Betti Structure

- Input from global geometry. Assume M has a MUM point at $t = 0$ and another conifold point at $t = p$. Choose a path σ from 0 to p with no self-intersections, and a basepoint $p_0 \in \sigma$ close to 0. Choose a loop starting at p_0, going out along σ, winding once around p, and then returning along σ to p. The Picard Lefschetz theorem says that the variation for a solution of L under this loop lies in a one dimensional space. We show this remains true for inhomogeneous solutions. Concretely, we think of solutions $\rho_k \in \mathbb{E}_n^{\vee,\nabla=0}$. We fix $\delta \in \mathbb{E}_n^{\nabla=0}$ such that a solution ρ is invariant around p if and only if $\langle \rho, \delta \rangle = 0$.

Betti Structure (cont.)

- We want to define a $\mathbb{Q}$-structure on the $\mathbb{C}$-Hodge variation $\mathbb{E}_n^\vee$. Assumptions:
  1. $\rho_0 \in \mathbb{E}_{n,\mathbb{Q}}^{\vee,\nabla=0}$; 2. The functional $\langle \bullet, \delta \rangle$ is $\mathbb{Q}$-rational, and $\langle \rho_0, \delta \rangle \neq 0$.
  3. The monodromy θ given by winding around 0 stabilizes the $\mathbb{Q}$-structure.
- Define $\kappa_k = \langle \rho_k, \delta \rangle / \langle \rho_0, \delta \rangle$. Generating series

$$\kappa(\varepsilon) := 1 + \kappa_1 \varepsilon + \kappa_2 \varepsilon^2 + \cdots$$

$$\alpha(\varepsilon) := 1/\kappa(\varepsilon) = 1 - \kappa_1 \varepsilon + (\kappa_1^2 - \kappa_2)\varepsilon^2 + \cdots$$

- $\mathbb{Q}$-basis given by $\eta_k := \rho_k + \alpha_1 \rho_{k-1} + \cdots + \alpha_k \rho_0$, $k \geq 0$.

Betti Structure (G-Z case)

- $\lambda(\varepsilon) := \log(\kappa(\varepsilon))$ ([], section 2.4; family of $K3$-surfaces Picard rank 19)

$$\lambda_1 = 0, \lambda_2 = -2\zeta(2), \lambda_3 = (17/6)\zeta(3), \lambda_4 = -3\zeta(4),$$
$$\lambda_5 = (7/3)\zeta(5), \lambda_6 = (-2/3)\zeta(6) - (1/72)\zeta(3)^2,$$
$$\lambda_7 = -(5/3)\zeta(7) + (1/6)\zeta(3)\zeta(4),$$
$$\lambda_8 = (29/12)\zeta(8) - (11/18)\zeta(3)\zeta(5),$$
$$\lambda_9 = (8/9)\zeta(9) + (5/3)\zeta(3)\zeta(6) + (11/3)\zeta(4)\zeta(5) + (17/648)\zeta(3)^3,$$
$$\lambda_{10} = -(147/5)\zeta(10) - (59/18)\zeta(3)\zeta(7) - (121/18)\zeta(5)^2 - (17/36)\zeta(4)\zeta(3)^2.$$

- Note some of the λ_j are computed numerically. λ_j for $j \geq 11$ involve multiple zeta values. For details see op. cit.

The Main Theorem

- The Taylor series of the motivic Γ-function associated to $h \in M$ as above is related to the κ-series for the dual connection. To avoid confusion, in what follows we write M^* instead of $M^\vee$ and we add a superscript * to indicate a quantity associated to M^*.
- For simplicity, assume monodromy at $t = 0$ is unipotent with one Jordan block (MUM condition). We assume also that the variation of ρ_0^* about the conifold point p is non-zero. We define $\kappa_0^* = 1$ and $\kappa_i^* \mathrm{Var}(\rho_0^*) = \mathrm{Var}(\rho_i^*)$. We write $\kappa^*(s) = \sum_{i \geq 0} k_i^* s^i$. Let $V \subset U$ be a tubular neighborhood of the path σ connecting 0 and p. We assume the closure of V contains $0, p$ but no other point of $\mathbb{P}^1 - U$. Then there exists an homology class $\xi \in H_1(V, \mathcal{M}^\vee)$ such that

$$\Gamma_\xi(s) = (e^{2\pi i s} - 1)^r \kappa^*(s).$$

- In the Golyshev-Zagier example, $M \cong M^*$.

The Main Theorem; Sketch of proof: Duality

- $\mathcal{D} = \mathcal{O}[D]$. Duality for holonomic $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}L$
  $L = q_0(t)D^r + \cdots + q_r(t)$.
- $(\mathcal{D}/\mathcal{D}L)^\vee \cong \mathcal{D}/\mathcal{D}L^\vee$
  $L^\vee := -(-D)^r q_0(t) - (-D)^{r-1} q_1(t) - \cdots - q_r(t)$
- Pairing on $\mathcal{O}$:

$$\{\phi, \psi\} := \sum_{k=1}^{r} \sum_{\substack{i,j \geq 0 \\ i+j=k-1}} (-D)^i (q_{r-k}\psi) D^j \phi$$

- $D\{\phi, \psi\} = (L\phi)\psi + \phi(L^\vee \psi)$

The Main Theorem; Sketch of proof: Integration by parts

- The discussion which follows is oversimplified. The full argument uses that $\pi_1(V)$ is free on 2 generators to understand classes $\xi \in H_1(V, \mathcal{M} \otimes t^s)$. Here I want only to exhibit the generalized integration by parts computation. I assume $\xi = \psi|_{\sigma_p}$, where σ_p loops once around the conifold point p.
- Take $\Phi \in \mathcal{O}[[s]]$ satisfying $L^\vee \Phi = s^r t^s$. (Φ is an inhomogeneous solution for $L^\vee$.)

$$\Gamma_\xi(s) = \int_{\sigma_p} t^s \psi(t) \frac{dt}{t} = \int_{\sigma_p} (L^\vee \Phi)\psi \frac{dt}{t} \tag{1}$$

$$= \int_{\sigma_p} D\{\Psi, \phi\} \frac{dt}{t} \tag{2}$$

$$= (\sigma_p - 1)\{\Phi, \psi\} \tag{3}$$

The Main Theorem; end of proof

- Remaining calculation (omitted) uses $(\sigma_p - 1)\Phi = \kappa^*(s)\delta^\vee$ with $\delta^\vee$ a solution for $L^\vee$.

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