Nov. 10, 2009

Dear Francis,

(The following theorem was proven by my student Eric Patterson. He has decided not to go into mathematics, so there was some delay in promulgating his thesis, prompting me to work out my own proof, given below. I stress, however, that the result was originally due to Eric. He promises to submit a paper for publication.)

Let $H = H_1(\Gamma)$ for a graph Γ, and write $X = X_\Gamma$ for the graph hypersurface. Recall X parametrizes a family of symmetric matrices (upto scale factor) $M_x = \sum x_e M_e$ such that $\dim\ker M_x \geq 1$. Let $X_p = \{x \in X \mid \dim\ker M_x > p\}$ so e.g. $X = X_0$. Let $X(p) \subset X$ be defined by the vanishing of all partial derivatives of order $\leq p$, so e.g. $X = X(0)$ and $X(1) = X_{sing}$, the singular points of X.

Theorem 1. *We have $X_i = X(i)$ for all i.*

Proof.

Lemma 2. *$X_p \subset X(p)$.*

proof of lemma. to an edge e of Γ we associate the functional $e^\vee : H_1(\Gamma) \to \mathbb{Q}$ which associates to a loop the multiplicity of e in the loop. We have $H_1(\Gamma - e) = \ker(e^\vee)$. If $e^\vee \neq 0$ then on graph polynomials we have $\Psi_{\Gamma-e} = \partial/\partial A_e \Psi_\Gamma$. What's more, the quadratic form associated to the symmetric matrix M_x, when restricted to $H_1(\Gamma - e)$ has determinant given upto nonzero scale by $\Psi_{\Gamma-e}$. Now let $e_1, \ldots, e_p$ be edges, and assume that the intersection $\bigcap \ker e_i^\vee$ is proper. Let $x \in X_p$. Then the nullspace of the quadratic form associated to M_x has dimension $\geq p+1$, so it cannot be the case that the quadratic form restricted to $\bigcap_{i=1}^p \ker e_i^\vee$ is nondegenerate. it follows that $\partial/\partial A_{e_1} \cdots \partial/\partial A_{e_p} \Psi(x) = 0$, so $x \in X(p)$. □

Lemma 3. *Let H be a finite dimensional vector space of dimension r over a field k of characteristic $\neq 2$. Let $e_i^\vee : H \to k$ be linear functionals, $1 \leq i \leq n$, and assume $\oplus e_i^\vee : H \hookrightarrow k^n$. Let $Q : H \to k$ be a quadratic form, and let $N \subset H$ be the nullspace of Q. Assume $\dim N = s > 0$. Then there exist $i_1, \ldots, i_s$ such that, writing $L = \bigcap \ker e_{i_j}^\vee \subset H$, we have $Q|L$ nondegenerate.*

proof of lemma. We fix a framing $H = k^r$ such that Q is diagonal; $Q = f_1 x_1^2 + \cdots + f_r x_r^2$. Let $B = (b_{ij})_{1 \leq i \leq r, 1 \leq j \leq r-s}$ be an $(r \times (r-s))$-matrix of maximal rank $r - s$ which we think of as an embedding

$k^{r-s} \hookrightarrow B(k^{r-s}) = L \subset k^r$, and hence a framing $L \cong k^{r-s}$. We identify k^r with its own dual in the standard way, so $Q : k^r \to (k^r)^\vee = k^r$ is given by the diagonal matrix with entries $f_1, \ldots, f_r$. Then in terms of the framing, $Q|L$ is given by the symmetric $((r-s) \times (r-s))$ matrix

$$C := {}^tB \begin{pmatrix} f_1 & 0 & 0 & \ldots \\ 0 & f_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & \ldots & f_r \end{pmatrix} B \tag{1}$$

The entries of C are linear forms in the f_i, so $D := \det(C)$ is homogeneous of degree $r-s$ in the f_i. If we specialize any $s+1$ of the f_i to 0, the determinant D dies, so D is necessarily a linear combination of monomials of degree $r-s$ in the f_i which have degree ≤ 1 in each f_i. To calculate the coefficient in D of one of those monomials, say $f_1 f_2 \cdots f_{r-s}$, we can drop the bottom s rows of B and the last s columns of ${}^tB$, getting

(2)

$$\det\left(\begin{pmatrix} b_{11} & b_{21} & \ldots & b_{r-s,1} \\ \vdots & \vdots & \ldots & \\ b_{1,r-s} & \ldots & \ldots & b_{r-s,r-s} \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 & \ldots \\ 0 & f_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & \ldots & f_{r-s} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1,r-s} \\ \vdots & \vdots & \ldots & \\ b_{r-s,1} & \ldots & \ldots & b_{r-s,r-s} \end{pmatrix} \right) =$$

$$\det \begin{pmatrix} b_{11} & b_{21} & \ldots & b_{r-s,1} \\ \vdots & \vdots & \ldots & \\ b_{1,r-s} & \ldots & \ldots & b_{r-s,r-s} \end{pmatrix}^2 f_1 f_2 \cdots f_{r-s}.$$

Let $e_i^\vee$ correspond to $(a_{i1}, \ldots, a_{ir}) \in (k^r)^\vee$. By assumption these vectors span $(k^r)^\vee$, and we need to show we can choose s of them so the perpendicular space L has a basis $(b_{ij})_{1 \leq i \leq r, 1 \leq j \leq r-s}$ such that the square submatrix $(b_{ij})_{1 \leq i \leq r-s, 1 \leq j \leq r-s}$ has nontrivial determinant. This is equivalent to requiring that the projection from L onto the first $r-s$ coordinates of $H = k^r$ is surjective. By dimension, this is equivalent to the projection being injective, i.e. L should not meet the span of the last s coordinate vectors. Given any $W \subsetneq H$, our hypotheses imply we can choose $e^\vee$ with $W \not\subset \ker(e^\vee)$. Start with $W = W_0$, the span of the last s coordinate vectors. Then take $W_1 = W_0 \cap \ker(e^\vee)$ so $\dim W_1 = s - 1$. Continuing in this way, we can finally write $(0) = W_0 \cap \bigcap_{i=1}^s \ker(e_i^\vee)$. □

To prove the theorem, we consider the diagram of inclusions from lemma 2

$$
\begin{array}{ccccccc}
X_0 & \overset{\supset}{\longleftarrow} & X_1 & \overset{\supset}{\longleftarrow} & X_2 & \overset{\supset}{\longleftarrow} & \cdots \\
\| & & \downarrow & & \downarrow & & \\
X(0) & \overset{\supset}{\longleftarrow} & X(1) & \overset{\supset}{\longleftarrow} & X(2) & \overset{\supset}{\longleftarrow} & \cdots
\end{array} \tag{3}
$$

As a consequence of lemma 3, it follows that

$$X_i - X_{i+1} \subset X(i) - X(i+1) \tag{4}$$

(I.e. if the nullspace has dimension exactly $i+1$, then there exists some $i+1$-st order partial which doesn't vanish.) Taking a disjoint union for $i = 0, 1, \ldots, j-1$ we get

$$X_0 - X_j \subset X(0) - X(j) \tag{5}$$

Since $X_0 = X(0)$, it follows that $X(j) \subset X_j$. Together with lemma 2, this completes the proof. □

Best,
Spencer