Math 259 - Final Problems List
June, 2003

(1) (a) State carefully and precisely the fundamental theorem of galois theory.
(b) Describe in full detail the structure of cyclotomic field extensions. Your description should include information about cyclotomic polynomials, galois groups of cyclotomic fields over $\mathbb{Q}$, and subfields of $\mathbb{Q}(\sqrt[p]{1})$ with generators for p prime.

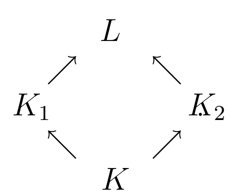
(2) (a) Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial of degree 3. Let $K/\mathbb{Q}$ be the splitting field of f. Discuss the various possibilities for $[K : \mathbb{Q}]$ and for $Gal(K/\mathbb{Q})$.
(b) Let $g \in \mathbb{Q}[X]$ be a polynomial and factor

$$g = (X - \alpha_1) \dots (X - \alpha_n)$$

with $\alpha_r \in \mathbb{C}$. Show that the number of α_r which do not lie in $\mathbb{R}$ is even.
(c) Suppose f in (a) has exactly one root in $\mathbb{R}$. What are the possibilities for $Gal(K/\mathbb{Q})$?

(3) (a) A field extension is *abelian* if it is galois with abelian galois group. Let K/k be an abelian extension. Let L be a field with $k \subset L \subset K$. Show L/k and K/L are abelian. State precisely any theorem you use.
(b) Now let K/k be *any* finite separable extension of degree n. Let $K = k(x)$ for some x satisfying an irreducible, separable polynomial $P(X) \in k[X]$ of degree n. Assume the *splitting field* F for P over k is abelian. Show that $K = F$.

(4) Consider a diagram of fields (with all extensions over K assumed separable)



Assume that L is the compositum of K_1 and K_2, i.e. that any subfield of L which contains K_1 and K_2 is equal to L. Suppose further that K_1/K is galois with group G. Show that L/K_2 is galois with galois group H for some subgroup $H \subset G$. (Hint: First use the fact that K_1/K is splitting to show that L/K_2 is splitting. Then show the galois group of L/K_2 stabilizes K_1, i.e. for $g \in \mathrm{Gal}(L/K_2)$ we have $g(K_1) = K_1$.)

(5) Consider the polynomial $X^4 + aX^2 + b \in K[X]$, where K is a field of characteristic $\neq 2$. Let L be the splitting field, and let $G = Gal(L/K)$ be the Galois group.
   (a) Show $G \subset \mathcal{S}_4$ where $\mathcal{S}_4$ is the permutation group on the 4 roots $\alpha_1, \ldots, \alpha_4$.
   (b) Write $(ij) \in \mathcal{S}_4$ for the transposition which interchanges α_i and α_j. Let $D \subset \mathcal{S}_4$ be the subgroup generated by $(13)(24)$ and (12) (i.e. $D = \langle (13)(24), (12) \rangle$). Show D has order 8.
   (c) Show that after possibly reordering the roots, we have $G \subset D$. Give an example where $G = D$.

(6) (a) Define the discriminant D of a polynomial $f \in K[X]$.
   (b) What can one conclude about the Galois theory of the splitting field of f over K if D is a square in K. (Assume characteristic of K is not 2.)
   (c) The discriminant of the polynomial $X^3 + pX + q$ is computed in the book to be $D = -4p^3 - 27q^2$. What can you say about the Galois group of the polynomial $X^3 - 3X + 1$ over $\mathbb{Q}$.

(7) Let K be a field and let t be a variable. Let $K(t)$ be the field of rational functions in t with coefficients in K.
   (a) Let $P(t) \in K[t]$ be a *polynomial* in t of degree d. Let $K(P) \subset K(t)$ be the subfield of rational functions in $P(t)$. (So $K(P)$ is the smallest subfield of $K(t)$ containing K and P.) Show that $[K(t) : K(P)] = d$. What is the minimal polynomial for t over $K(P)$?
   (b) Let $R(t) = P(t)/Q(t)$ be a rational function. Give a monic polynomial satisfied by t over $K(R)$. You need not show this polynomial is of minimal degree.