Math 270 - Practice Midterm
April 22, 2004

(1) (a) let $f : \mathbb{C} \to \mathbb{C}$ be a function. State the Cauchy Riemann equations for f.
(b) Define analyticity and show that if f is analytic then f satisfies the Cauchy Riemann equations.
(c) Suppose that f is real-valued, i.e. has image in $\mathbb{R} \subset \mathbb{C}$. Show that f is analytic if and only if f is constant.

(2) (a) Define $|z|$, $\arg z$, $Re\ z$, $\overline{z}$, e^z.
(b) Compute $|e^z|$ and $\arg e^z$.
(c) Suppose $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ where u and v are real-valued. Express $f(\overline{z})$ (f of the complex conjugate of z) and $\overline{f(z)}$ (the complex conjugate of $f(z)$) in terms of u, v, x, y. Show that $\overline{f(\overline{z})}$ is analytic if f is.

(3) (a) Let $\gamma : [0, 1] \to \mathbb{C}$ be a path, and let f be a function on $\mathbb{C}$. Define the path integral

$$\int_\gamma f(z)dz.$$

(b) Discuss Cauchy's theorem and Cauchy's integral formula. Your discussion should include statements of the results and a brief, intuitive explanation of how to deduce the integral formula from the theorem.
(c) Compute

$$\int_{\{|z|=3\}} \frac{(z^2+7)dz}{z-1}$$

(4) (a) Let f be defined on an open set $U \subset \mathbb{C}$. Suppose for some $z_0 \in U$ that $f(z_0) \neq 0$. Show f is analytic at z_0 if and only if $g(z) = 1/f(z)$ is analytic at z_0.
(b) A *linear fractional transformation* is a function $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$. We assume c, d are not both 0. Show by directly computing the limit that f is analytic at any point $z \neq -d/c$. Show that $1/f(z)$ is analytic at $-d/c$.
(c) Let $h(z) = e^{1/z}$. Show that neither $h(z)$ nor $1/h(z)$ is analytic at $z = 0$.

(5) (a) Describe briefly the definition and basic properties of the Taylor series expansion of an analytic function f at a point z_0.

(b) Compute the first 3 terms in the Taylor series for $f(z) = \sqrt{z}$ around the point $z = 1$. (Take the branch $f(1) = 1$.)

(c) Suppose f is analytic at z_0 and $f(z_0) = 0$. Show $g(z) = \frac{f(z)}{z - z_0}$ is analytic at z_0. What is the Taylor series expansion for g at z_0?