Motivic Hopf Algebras and Periods Associated to Multiple Zeta Functions and Feynman Graphs

Lecture I: Work of Goncharov (+Deligne, Manin) on multiple zetas

Lecture II: Joint work with H. Esnault and D. Kreimer on Feynman graphs

1. Hopf algebra

$$\varinjlim \left(\mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, *)]/I^N\right)^\vee$$

in category of mixed Tate motives over $\mathbb{Z}$

2. Renormalization Hopf algebra in the category of motives.

3. Periods

Comments

1. Compare *and contrast* the two Hopf algebras.

2. Problem: what is the physical meaning of co-representations of the renormalization Hopf algebra. Link with Dyson Schwinger equations.

3. Fantasy: The action of the renormalization group on the the renormalization hopf algebra in physics can in some way be related to the action of the Tannaka group of the category of motives.

Tannakian Categories

Tannakian category $\mathcal{T}$ abelian, $\mathbb{Q}$-linear category with $\otimes$ (symmetric monoidal category).

$\mathcal{T}$ *rigid tensor category* (Identity object $\mathbf{1}$ with $X \otimes \mathbf{1} \cong \mathbf{X}$ and internal Hom $\underline{Hom}(X, Y)$.)

$F : \mathcal{T} \longrightarrow \{$f.d. $\mathbb{Q}$-vector spaces$\}$ *fibre functor* (exact, faithful, compatible with $\otimes$)

Example: $G = \mathrm{Spec}\,(A)$ affine algebraic group (A = commutative Hopf algebra).

$\mathcal{T} = Rep_k(G)$ category of f.d. representations of G over a field k. (To fix ideas, $k = \mathbb{Q}$)

$F(V) = V$ forgetful functor

Tannakian Categories
Main Theorem

Main Theorem. Can recover G from $(\mathcal{T},\ F)$.
$G = Aut^{\otimes} F$.

Example: $\mathcal{T} =$ f.d. $\mathbb{Z}$-graded vector spaces;

$F : \mathcal{T} \to$ f.d. vector spaces (forget grading.)

$G = \mathbb{G}_m$. $g \in \mathbb{G}_m(\mathbb{Q}) = \mathbb{Q}^\times$,
$\mathbb{Q} = F(\mathbb{Q}_n) \xrightarrow{g^n} F(\mathbb{Q}_n) = \mathbb{Q}$.

Unipotent Algebraic Groups

Assume:

1. $\mathcal{T}$ Artinian, simple objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

$$Hom(\mathbb{Q}(n), \mathbb{Q}(m)) = \mathbb{Q} \cdot \delta_{m,n}$$

2. Objects have weight filtrations $W_\bullet X$ with $gr^W_{2n} X = \oplus Q(-n)$.

Graded fibre functor

$$F_{gr} : \mathcal{T} \to \text{f.d. graded vector sp.}$$

$$F_{gr}(M) := \oplus_n Hom_{\mathcal{T}}(\mathbb{Q}(n), gr^W_{-2n} M)$$

$F :=$ forget $\circ F_{gr}$.

In this case $G = U \rtimes \mathbb{G}_m$ with U unipotent. (U acts as identity on gr^W_*)

Motivic Hopf Algebra Assoc. to π_1

Voevodsky's motivic category is a triangulated category, not a Tannakian category.

In nice cases (number fields) we do know how to build Tannakian categories.

Think of objects in our Tannakian category as being generalized cohomology groups.

Ex: X variety, $a \in X(k)$ point, $n > 1$,
$Y_i \subset X^n,\ 0 \leq i \leq n$

$Y_0 = \{a\} \times X^{n-1},\ Y_1 = \Delta_X \times X^{n-2}$,
$Y_2 = X \times \Delta_X \times X^{n-3}, \ldots, Y_n = X^{n-1} \times \{a\}$
Theorem(Beilinson).
$H^n(X^n, Y; \mathbb{Q}) \cong \mathrm{coker}(\mathbb{Q} \to \mathbb{Q}[\pi_1(X,a)]/I^n)^\vee$.
Here $I \subset \mathbb{Q}[\pi_1]$ is the augmentation ideal.

This shows motivic nature of nilpotent quotients of π_1.

Motivic Hopf Algebra Assoc. to π_1 (cont.)

n-step Nilpotent representations of $\pi_1 \leftrightarrow$ representations of the algebraic group $\mathrm{Spec}\,((\varinjlim \mathbb{Q}[\pi_1]/I^n)^\vee)$

$(\varinjlim \mathbb{Q}[\pi_1]/I^n)^\vee$ is a commutative Hopf algebra in the Tannakian category of motives.

More generally, $a, b \in X(k)$, can define the dual to the paths $(\varinjlim \mathbb{Q}[{}_aP_b]/I^n)^\vee$ is a ring object in the Tannakian category, so can define the torsor of paths $\mathrm{Spec}\,((\varinjlim \mathbb{Q}[{}_aP_b]/I^n)^\vee)$

Fibre Functors for Mixed Tate Motives

$\mathcal{T}$ Tannakian category of mixed Tate Motives. 3 fibre functors:

$F_\omega(M) := \oplus_n Hom_{\mathcal{T}}(\mathbb{Q}(n), gr^W_{-2n}M)$

$F_{Betti}(M) := H_{Betti}(M)$. (NB. in general, depends on embedding $\sigma : k \hookrightarrow \mathbb{C}$. We take $k = \mathbb{Q}$)

$F_{DR}(M) = H_{DR}(M)$ de Rham cohomology.

Comparison isomorphisms
$comp_{Betti,\omega} : M_\omega \otimes \mathbb{C} \xrightarrow{\cong} M_{Betti} \otimes \mathbb{C}$.

Similarly $comp_{DR,Betti}$, etc.

NB. $Isom(F_\omega, F_{Betti})$ is a $G = Gal(\mathcal{T})$-torsor.

$G = U \rtimes \mathbb{G}_m$ has no non-split torsors.

$$\therefore \exists\ a \in G(\mathbb{C}),$$
$$M_{Betti} = comp_{Betti,\omega}(a(M_\omega)) \subset M_{Betti} \otimes \mathbb{C}$$

DR realization: cosimplicial scheme

$\partial_i : X^{n-1} \to X^n$

$$(x_1, \ldots, x_{n-1}) \mapsto$$

$$\begin{cases} (a, x_1, \ldots, x_{n-1}) & i = 0 \\ (x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_{n-1}) & 1 \le i \le n-1 \\ (x_1, \ldots, x_{n-1}, a) & i = n \end{cases}$$

For X affine, resulting DR complex is quasi-isomorphic to Chen's complex of iterated integrals.

$n \mapsto \otimes_n \Gamma(X, \Omega^*)$; $\quad d\langle a \rangle = \langle da \rangle + (-1)^p \langle \sum (\pm) \partial_i^* a \rangle$

Ex: $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$.

$\mathbb{P}^1 - \{0, 1, \infty\}$; dch

Tangential basepoint at a puncture:
Intuitively, a tangent vector at a puncture doesn't determine a point of $\mathbb{P}^1 - \{0, 1, \infty\}$, but it does determine the fibre of a covering upto canonical identification, i.e. it defines a functor from coverings to sets. Thus it can be used as a basepoint for π_1.

$dch \in ({}_{0\to 1}P_{1\to 0})_{Betti}$ (path from 0 to 1 with tangential basepoints)

Periods: Mixed Tate Hodge and de Rham Structures

$n \in \mathbb{Z}$, define 1 dim. $\mathbb{Q}$-vect. spaces $\mathbb{Q}(n)_{Betti}$, $\mathbb{Q}(n)_{DR}$ weight $-2n$, $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$.

$$\mathbb{Q}(n)_{DR} \otimes \mathbb{C} = \mathbb{Q}(n)_{Betti} \otimes \mathbb{C} = H^{-n,-n}(\mathbb{Q}(n))$$

(Hodge filtration)

General Mixed Tate Hodge structure H; successive extension of $\mathbb{Q}(n)$'s.

$W_* H$ increasing weight filtration;
$F^* H_{\mathbb{C}}$ decreasing Hodge filtration

H mixed Tate if $gr^W H = \oplus_i \mathbb{Q}(n_i)$.

In this case $H_{\mathbb{C}} = W_{2n} H_{\mathbb{C}} \oplus F^{n+1} H_{\mathbb{C}}$

Periods: Example

$\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times = \mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}$

$\mathbb{Q} \cdot S^1 = H_1(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}(1)_{Betti}$;

$H^1(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}(-1)$;

$Q(-1)_{DR} = \mathbb{Q} \cdot \frac{dt}{t}$; $\mathbb{Q}(-1)_{Betti} = \mathbb{Q} \cdot (S^1)^\vee$.

$$Q(-1)_{DR} = 2\pi i \cdot \mathbb{Q}(-1)_{Betti} \subset \mathbb{Q}(-1)_{Betti} \otimes \mathbb{C} = \mathbb{Q}(-1)_{DR} \otimes \mathbb{C}$$

$\mathbb{Q}(-n)_{DR} = (2\pi i)^n \cdot \mathbb{Q}(-n)_{Betti}$

Periods: Typical computation

$\mathbb{P}^n$, homogeneous coordinates $A_0, \ldots, A_n$

$\Delta : \prod A_i = 0$ coordinate simplex; $X \subset \mathbb{P}^n$ hypersurface

Assume vertices $(0, \ldots, 1, \ldots, 0) \notin X$; define $H := H^n(\mathbb{P}^n - X, \Delta - X \cap \Delta)$

H is a Hodge structure (not necessarily mixed Tate).

Lowest weight $W_0 H = \mathbb{Q}(0)$ carried by the vertices.

Homology: $H^\vee \twoheadrightarrow \mathbb{Q}(0)$. Chose *splitting* $\mathbb{Q} \hookrightarrow H^\vee$

Ex: $\sigma = \{(\sigma_0, \ldots, \sigma_n) \mid \sigma_i \geq 0\} \subset \mathbb{P}^n(\mathbb{R})$

If $\sigma \cap X = \emptyset$, then $[\sigma] \in H^\vee = H_n(\mathbb{P}^n - X, \Delta - X \cap \Delta)$ gives splitting.

Computation (cont.)

Relative DR-complex. $V \hookrightarrow W$ closed.

$\Omega^*_{W,V} \subset \Omega^*_W$.

Ex: $V \subset \mathbb{P}^1$ finite set of points. $I \subset \mathcal{O}_{\mathbb{P}^1}$ ideal

Note $n = \dim W$, $\Omega^n_{W,V} = \Omega^n_W$. For W affine, get

$\Gamma(W, \Omega^n) \to H^n_{DR}(W, V)$

$W = \mathbb{P}^n - X,\ X : F = 0,\ \deg F = d;$

$\Omega_n = \sum(\pm) A_i dA_0 \ldots dA_i \ldots dA_n,\ \omega = \frac{G\Omega_n}{F^r}$

log divergent Feynman graph case; G graph, ℓ loops, $n+1 = 2\ell$ edges

F graph polynomial, $\omega_G = \frac{\Omega_n}{F^2}$

Period $= \int_\sigma \omega_G$

Multiple Zeta Periods

$$\sigma = \{(x_1, \ldots, x_n) \in [0,1]^n \mid 0 \le x_1 \le \ldots \le x_n \le 1\} \subset (\mathbb{P}^1)^n$$

$pr_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$

$$\omega := pr_1^*(\frac{dt}{t - \varepsilon_1}) \wedge \ldots \wedge pr_n^*(\frac{dt}{t - \varepsilon_n}); \quad \varepsilon_i = 0,\ 1$$

Important theme: polar locus
$X = (\mathbb{P}^1)^n - \prod_i(\mathbb{P}^1 - \{\varepsilon_i, \infty\})$ meets σ.

Must blow up. Correct base is not $(\mathbb{P}^1)^n$ but rather $\overline{M}_{0,n+3} \to (\mathbb{P}^1)^n$.

Moduli space of semi-stable genus 0 curves (Goncharov-Manin)

Discussion of $\overline{M}_{0,n}$

$\overline{M}_{0,n}$ moduli space of *semi-stable* genus 0 curves.

Tree of $\mathbb{P}^1$'s (connected, no loops) with only ordinary double points and with n smooth points marked.

Group of automorphisms required to be finite ($\Leftrightarrow$ number of singular + marked points ≥ 3 on every irreducible component.)

Fine moduli space; $\mathcal{C}_n \xrightarrow{p_n} \overline{M}_{0,n}$ universal curve with n marked points.

Remarkable fact:

$$\begin{tikzcd} \mathcal{C}_n \arrow[rr, phantom, "\cong"] \arrow[dr] & & \overline{M}_{0,n+1} \arrow[dl, "\text{forget } n+1\text{-st pt.}"] \\ & \overline{M}_{0,n} & \end{tikzcd}$$

$\overline{M}_{0,5}$

Ex. $\overline{M}_{0,4} \cong \mathbb{P}^1$ (crossratio). $\overline{M}_{0,5} = \mathbb{P}^1 \times \mathbb{P}^1$ with $(0,0),\ (1,1),\ (\infty,\infty)$ blown up.

Note that chain σ on $\mathbb{P}^1 \times \mathbb{P}^1$ meets polar locus for $\frac{dt_1}{t_1-1} \circ \frac{dt_2}{t_2}$ at points $(0,0),\ (1,1)$. These points are blown up on $\overline{M}_{0,5}$.

Iterated Integrals

$\sigma \hookrightarrow [0,1]^n \hookrightarrow (\mathbb{P}^1)^n$

$\int_0^1 \omega_1 \circ \ldots \circ \omega_n := \int_\sigma pr_1^* \omega_1 \wedge \ldots \wedge pr_n^* \omega_n$

$$\underbrace{\frac{dt}{t-1} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_1} \circ \ldots \circ \underbrace{\frac{dt}{t-1} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_r}$$

power series expansions for $\frac{1}{t-1}$ yield multiple zeta value

$$\text{Period} = \sum_{a_r > \ldots > a_1 > 0} \frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}}$$

Deligne-Goncharov Construction

$(\varinjlim \mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \text{tang. bpt.})]/I^N)^\vee \in Ob(\mathcal{T})$

Action of $G = Gal(\mathcal{T})$ on this object yields $\iota : G_\omega \to H_\omega \subset \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$

Measures linear algebra of G acting on $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \text{tang. bpt.})_{mot}$.

Linear algebra identifies $H_\omega = V_\omega \rtimes \mathbb{G}_m$ and $V_\omega(\mathbb{C}) = \prod :=$ grouplike elts. in $\mathbb{C}\langle\langle e_0, e_1\rangle\rangle$

In particular, $\langle dch\rangle \in H_\omega(\mathbb{C})$

$\mathbb{Q}$-form $({}_1P_0)_{Betti} \subset ({}_1P_0)_\omega \otimes \mathbb{C}$ equals image of $\mathbb{Q}$-form $({}_1P_0)_\omega$ by $\langle dch\rangle \cdot \tau(2\pi i)$.

Consequence: $\langle dch\rangle \cdot \tau(2\pi i) = \iota(a) \cdot v; \quad v \in H_\omega(\mathbb{Q})$

I.e.

$$\tau(2\pi i)^{-1}\langle dch\rangle \in \iota(U_\omega) \cdot v \subset V_\omega$$

Deligne-Goncharov Construction (cont.)

Coefficient of $e_{i_1} \cdots e_{i_n}$ in $\langle dch \rangle$ is the corresponding iterated integral, $e_{i_j} = 0,\ 1$

This is the link between multiple zetas and the structure of $G = Gal(\mathcal{T})$

Theorem. Let D_n be the dim. of $\mathbb{Q}$-vector space generated by coefficients of dch in degree n. Then generating series $\sum D_n t^n$ is term by term bounded by the coefficients of $\frac{1}{1-t^2-t^3}$.

(Also proven by Terasoma).

Graph Hopf Algebra

Hopf algebra of graphs. (Kreimer, Connes-Kreimer)

$\mathcal{H} = \oplus \mathbb{Q} \cdot \Gamma$ vector space on isomorphism classes of graphs

Commutative multiplication: $\Gamma \cdot \Gamma' := \Gamma \amalg \Gamma'$

Comultiplication $\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H}$

$$\Delta(\Gamma) = \sum_{\substack{\Gamma' \subset \Gamma \\ \text{divergent subgraph}}} \Gamma' \otimes (\Gamma /\!\!/ \Gamma')$$

($\Gamma /\!\!/ \Gamma'$ means the quotient graph with components of Γ' shrunk to points.)

The precise meaning of "divergent subgraph" in the Kreimer picture is determined by the physical picture (the Lagrangian)

One has to be careful that the co-multiplication be co-associative.

Defn. $\Gamma' \subset \Gamma$ divergent, if $2h_1(\Gamma') \geq \#(Edge(\Gamma')$

Motive Associated to a Graph

Construct $M(\Gamma)$ so that Feynman amplitude for Γ is a period.

Γ with n loops, N edges.

Perturbative series leads to a certain integral of the form

$$F.A.(\Gamma) = \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{P\Omega_{4n-1}}{Q_1 Q_2 \cdots Q_N}$$

Here Q_i rank 4 quadrics (≥ 0), and P homogeneous of degree $2N - 4n$.

$\Omega = \sum_{i=1}^{4n} \pm B_i dB_1 \wedge \ldots \widehat{dB_i} \wedge \ldots \wedge dB_{4n}$

Schwinger Picture

$$\begin{array}{ccccc} \mathcal{Q}: \sum A_i Q_i = 0 & \hookrightarrow & \mathbb{P}^{N-1} \times \mathbb{P}^{4n-1} & \to & \mathbb{P}^{4n-1} \\ & \searrow f & \downarrow & & \\ & & \mathbb{P}^{N-1} & & \end{array}$$

f family of projective quadrics of dim $4n-2$

$Q_i \leftrightarrow \mathcal{M}_i,\ 4n \times 4n$ symmetric matrix

$\mathcal{M}_i = M_i \otimes I_4,\ M_i\ n \times n$ symmetric

Singular locus of f defined by
$\Phi(A) = \det(\sum A_i \mathcal{M}_i) = 0$

$\Phi(A) = \Psi(A)^4;\ \Psi(A) = \det(\sum A_i M_i)$

Schwinger Picture (cont.)

Fibres of f are even dim. quadrics $\mathcal{Q}_a$

Middle dim. primitive cohomology
$H^{4n-2}(\mathcal{Q}_a) \cong \mathbb{Q}$

rk 1 local system, monodromy $\{\pm 1\}$, ramified over $\{\Phi = 0\}$.

Cohomology of resulting double cover of $\mathbb{P}^{N-1}$

$$\frac{G\Omega_{N-1}}{\sqrt{\Phi}} = \frac{G\Omega_{N-1}}{\Psi^2}$$

Important case: $N = 2n,\ G = 1$.

$$F.A. = \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}}{Q_1 \cdots Q_{2n}} = \frac{1}{\pi^{2n}} \int_\sigma \frac{\Omega_{2n-1}}{\Psi^2}$$

$\sigma = \{(a_1, \ldots, a_{2n}) \in \mathbb{P}^{2n-1}(\mathbb{R}) \mid a_i \geq 0\}$

Graph Polynomial

$\mathbb{P}^{N-1} \supset X_\Gamma : \Psi = \Psi_\Gamma = 0;$

Facts about $\Psi_\Gamma(A)$:

1. $\Psi = \sum_{\substack{T\subset\Gamma \\ \text{span. tree}}} \prod_{e\notin T} A_e$

2. Γ connected. View Γ as metrized graph with e having “length” A_e and vertices orthonormal. Then get relation with Laplacian of graph.

$$\mathbb{Z}^{Edge(\Gamma)} \xrightarrow{d} \mathbb{Z}^{Vert(\Gamma)}; \quad \Psi = \prod A_e \cdot \det(dd^*)$$

3. $L : A_{e_1} = \ldots = A_{e_p} = 0$ coordinate linear space. Then

$$L \subset X_\Psi \Leftrightarrow h_1(e_1 \cup \ldots \cup e_p) > 0$$

$M(\Gamma)$

First try: $M(\Gamma) = H^{N-1}(\mathbb{P}^{N-1} - X_\Gamma)(N-1)$

Criticism: $\int_\sigma \frac{\Omega_{2n-1}}{\Psi^2}$ not defined because σ is a relative chain.

try again:

$$M(\Gamma) =$$
$$H^{N-1}(\mathbb{P}^{N-1} - X_\Gamma, \Delta - X_\Gamma \cap \Delta)(N-1)$$

$\Delta : \prod A_e = 0$ coordinate simplex. Criticism: period still not defined.

$$\sigma \cap X_\Gamma = \bigcup_{L \subset X_\Gamma} L(\mathbb{R})^+$$

Have to blow up to separate σ from X.

$$M(\Gamma) = H^{N-1}(P - Y, B - Y \cap B)$$
$$\rho : P \to \mathbb{P}^{N-1},\ B = \rho^{-1}(\Delta)$$

Example. Wheel with n-spokes WS_n. Hélène will talk about the structure of $M(WS_n) = H^{2n-1}(\mathbb{P}^{2n-1} - X_n)(2n-1)$

Algebra Structure

$\mathcal{H}_{mot} := \oplus_\Gamma M(\Gamma)$ (ind-motive)

Product structure: A_e edge variables for Γ, B_e edge variables for Λ.

$\Psi_{\Gamma \amalg \Lambda} = \Psi_\Gamma(A) \cdot \Psi_\Lambda(B)$

$\#\mathrm{Edge}(\Gamma) = N, \ \#\mathrm{Edge}(\Lambda) = S$

$X_{\Gamma\amalg\Lambda} \subset \mathbb{P}^{N+S-1}$

$$\mathbb{P}^{N+S-1} - X_{\Gamma\amalg\Lambda} \xrightarrow{h} (\mathbb{P}^{N-1} - X_\Gamma) \times (\mathbb{P}^{S-1} - X_\Lambda)$$

Here $h(A, B) = (A) \times (B)$.

h is a $\mathbb{G}_m$-fibration, $h(cA, c'B) = h(A, B)$.

$$\therefore H^{N+S-1}(\mathbb{P}^{N+S-1} - X_{\Gamma\amalg\Lambda}, \ldots)(N+S-1) \cong$$
$$H^{N-1}(\mathbb{P}^{N-1} - X_\Gamma, \ldots)(N-1) \otimes$$
$$H^{S-1}(\mathbb{P}^{S-1} - X_\Lambda, \ldots)(S-1)$$

$$M(\Gamma) \otimes M(\Lambda) \cong M(\Gamma \amalg \Lambda);$$

$$\mathcal{H}_{mot} \otimes \mathcal{H}_{mot} \to \mathcal{H}_{mot}$$

Co-algebra Structure

$\Lambda \subset \Gamma$ graphs.

$\Lambda = e_1 \cup \ldots \cup e_p; \quad L : A_1 = \ldots = A_p = 0.$

Assume $h_1(\Lambda) > 0$ so $L \subset X_\Gamma$.

Theorem. The normal cone of $L \subset X_\Gamma$ is a union of the normal cones of X_Λ and $X_{\Gamma /\!\!/ \Lambda}$. I.e.

$$\Psi_\Gamma = \Psi_\Lambda(A_1, \ldots, A_p) \cdot \Psi_{\Gamma /\!\!/ \Lambda} + R$$

where R has degree $> \deg \Psi_\Lambda$ in $A_1, \ldots, A_p$.

Reinterpretation. Consider the diagram

$$\begin{array}{ccc} Y \cup E & \longrightarrow & P \\ \Big\downarrow & & \Big\downarrow \pi \\ X & \longrightarrow & \mathbb{P}^{N-1} \end{array} \qquad (1)$$

Here π is the blowup of L in $\mathbb{P}^{N-1}$, Y is the strict transform of X, and $E \cong L \times \mathbb{P}^{p-1}$ is the exceptional divisor. Then

$$Y \cap E = (L \times X_\Lambda) \cup (X_{\Gamma /\!\!/ \Lambda} \times \mathbb{P}^{p-1}).$$

Co-algebra Structure (cont.)

Corollary. There is a well-defined residue map

$$M(\Gamma) \xrightarrow{Res} M(\Lambda) \otimes M(\Gamma /\!\!/ \Lambda)$$

Proof. $P := Blow(L \subset \mathbb{P}^{N-1})$.

$$\begin{aligned} H^{N-1}(P-Y)(N-1) &\to H^{N-1}(P-Y-E)(N-1) \\ &\xrightarrow{Res} H^{N-2}(E - Y \cap E)(N-2) \end{aligned}$$

But

$$M(\Gamma) \cong H^{N-1}(P - Y - E)(N - 1)$$

and

$$H^{N-2}(E - Y \cap E)(N - 2) \cong M(\Lambda) \otimes M(\Gamma /\!\!/ \Lambda)$$

QED.

Co-associativity

Must give a rule for which $\Lambda \subset \Gamma$ will occur in the comultiplication. Always include Γ, $\emptyset$. Note $M(\emptyset) = \mathbb{Q}(0) = \mathbf{1}$.

Co-associativity associated to renormalization: $\Lambda \subset \Gamma$ such that $2h_1(\Lambda) \geq \#\mathrm{Edge}(\Lambda)$.

Minimal subgraphs: $\Lambda \subset \Gamma$ such that $h_1(\Lambda) > 0$ and for any $\Xi \subsetneq \Lambda$ have $h_1(\Xi) < h_1(\Lambda)$.

The Category of Graphs?

Objects graphs, morphisms $\Gamma \to \Gamma /\!\!/ \Lambda$ when $h_1(\Lambda) = 0$ (subforests).

Link to $H^*(M_{g,n})$ (Kontsevich)

What about $\Gamma \to \Lambda \times \Gamma /\!\!/ \Lambda$ when $h_1(\Lambda) > 0$?

Link to $H^*(\overline{M}_{g,n})$??

Remark. One can show (collaboration with Kreimer and K. Yeats) that if one works with oriented graphs, the differential $\Gamma \mapsto \sum_{e \in \Gamma} \pm \Gamma/e$ in graph cohomology gives $\mathcal{H}$ the structure of differential Hopf algebra.

The Period

Renormalization theory (Feynman rules): homomorphism of algebras (not co-algebras)

$$\phi : \mathcal{H} \to \mathbb{C}[\log q]$$

On primitive graphs ($N = 2n$, no divergent subgraphs) this is our period

$$\int_\sigma \frac{\Omega_{2n-1}}{\Psi_\Gamma^2}.$$

Broadhurst-Kreimer: $WS_n \rightsquigarrow (*)\zeta(2n-3)$, etc.

The Hodge Structure

$N = 2n$. Recall $H^{N-1}_{Betti}(\mathbb{P}^{N-1} - X_\Gamma)$ has a Hodge filtration.

Question: $\omega_\Gamma := \frac{\Omega_{2n-1}}{\Psi_\Gamma^2} \in F^? H^{N-1}_{Betti}(\mathbb{P}^{N-1} - X_\Gamma)$.

For X smooth, would get $\omega_\Gamma \in F^{2n-2}$.

In WS_n case, by Broadhurst-Kreimer period $\in \mathbb{Q}^\times \cdot \zeta(2n-3)$. Expect an extension

$$0 \to \mathbb{Q}(0) \to E \to \mathbb{Q}(3-2n) \to 0$$

Recall $F^{2n-3}\mathbb{C}(3-2n) = \mathbb{C}$, so in this case expect $\omega_\Gamma \in F^{2n-3}$

Proposition. $\omega_\Gamma \notin F^{2n-2}$ for any Γ.

proof. $X : \det(T) = 0$, T $n \times n$ symmetric.
$Z \subset \mathbb{P}^{2n-1}$ locus where $\operatorname{rank}(T) \leq n-2$.

Then codim. Z in $\mathbb{P}^{2n-1}$ is ≤ 3. (Think of 2×2 case.

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ rk. 0 def. by $a = b = c = 0$.)

The Hodge Structure (cont.)

Coordinates z_1, z_2, z_3 such that $\psi_\Gamma \in (z_1, z_2, z_3)^2$.

Locally, if we blow up Z,

$$\omega_\Gamma = \frac{z_1^2 dz_1 \wedge d(z_2/z_1) \wedge d(z_3/z_1) \ldots}{z_1^4 \phi(z_1, z_2/z_1, z_3/z_1, \ldots)^2}$$

Pole of order 2 along exceptional divisor.

Using Deligne's pole order description of the Hodge filtration, we can conclude $\omega_\Gamma \notin F^{2n-2}$.

Remark. Results of Broadhurst-Kreimer suggest that sometimes $\omega_\Gamma \notin F^{2n-3}$.

Ex. Γ bipartite $(3,4)$ graph (all lines joining a point $\{a_1, a_2, a_3\}$ to a point $\{b_1, b_2, b_3, b_4\}$.) Γ has 12 edges and 6 loops. Period $= (*)\zeta(3,5)$. Thus expect $\omega_\Gamma \in F^8 H^{11}(\mathbb{P}^{11} - X_\Gamma)$. In this case, $n = 6,\ 8 = 2n - 4$.