Motivic Cohomology

1. Triangulated Category of Motives (Voevodsky)

2. Motivic Cohomology (Suslin-Voevodsky)

3. Higher Chow complexes

a. Arithmetic (Conjectures of Soulé and Fontaine, Perrin-Riou)

b. Mixed Tate Motives (Relations with motives associated to $\pi_1(\mathbb{P}^1 - S)$; Deligne, Goncharov...)

c. Limiting mixed motives.

Voevodsky's Construction Important Concepts (Mazza, Voevodsky, Weibel)

1. Additive category of correspondences Cor_k.
Objects smooth varieties over k.

Morphisms:

$$\begin{array}{ccc} & Z_i & \\ f_i \swarrow & & \searrow \\ X & & Y \end{array}$$

Here f_i finite surjective, and

$$Z = \sum n_i Z_i \in Hom_{Cor}(X, Y).$$

Example: $Z \subset X \times Y$ graph of $f : X \to Y$,

$$Sm_k \to Cor_k.$$

2. Presheaves with transfers **PST**:

$$F : Cor_k^{op} \to Ab$$

Examples of presheaves with transfer

1. Representable $\mathbb{Z}_{tr}(X)$, $X \in Ob(Sm_k)$.

$$\mathbb{Z}_{tr}(X)(U) := Hom_{Cor}(U, X).$$

2. $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$

$$\text{Coker}\Big(\bigoplus \mathbb{Z}_{tr}(X_1 \times \cdots \widehat{X_i} \times \cdots \times X_n) \to \mathbb{Z}_{tr}(\prod X_i) \Big)$$

3. $\mathbb{Z}_{tr}(\bigwedge^n \mathbb{G}_m)$. Take $X_i = \mathbb{A}^1 - \{0\}$, $x_i = 1$.

Chains, Homotopy Invariance

$$\Delta^n = \operatorname{Spec} k[x_0, \ldots, x_n]/(\sum x_i - 1)$$
$$\partial_i : \Delta^{n-1} \hookrightarrow \Delta^n$$

For $F \in Ob(\mathbf{PST})$

$$C_*(F)(U) := \ldots \to F(U \times \Delta^2) \to F(U \times \Delta^1) \to F(U).$$

F is *homotopy invariant* if

$$i_0^* = i_1^* : F(U \times \mathbb{A}^1) \to F(U).$$

Lemma 1 *In general,* $i_0^* = i_1^* : C_*F(U \times \mathbb{A}^1) \to C_*F(U)$ *are chain homotopic.*

Corollary 2 *The homology presheaves* $H_n(C_*F)$ *are homotopy invariant.*

$\mathbb{A}^1$-Homotopy

$f, g : X \to Y$ in Cor_k. $H : X \times \mathbb{A}^1 \to Y$.

$$H|X \times 0 = f; \quad H|X \times 1 = g$$

Equivalence relation in Cor. $f \simeq g$.

Proposition 3 *$f \simeq g$ Then f_*, g_* chain homotopy maps $C_*\mathbb{Z}_{tr}(X) \to C_*\mathbb{Z}_{tr}(Y)$.*

Motivic Cohomology

$$\mathbb{Z}(q) := C_*\mathbb{Z}_{tr}(\bigwedge^q \mathbb{G}_m)[-q]; \quad q \geq 0.$$

Complex of Zariski sheaves on X for any X.

Definition 4 *The motivic cohomology groups*

$$H^{p,q}(X) := H^p_{Zar}(X, \mathbb{Z}(q)).$$

Products: $\mathbb{Z}(q) \otimes \mathbb{Z}(s) \to \mathbb{Z}(q+s)$,
$H^{p,q}(X) \otimes H^{r,s}(X) \to H^{p+r,q+s}(X)$.

Low degrees:
$H^{0,0}(X) = \mathbb{Z}(\pi_0(X))$, $H^{p,0} = 0$; $p \geq 1$.
$\mathbb{Z}(1) \cong \mathcal{O}^*[-1]$.

Relation to Milnor K-theory:

Theorem 5 $H^{n,n}(k) \cong K_n^{Milnor}(k)$.

Triangulated Category of Motives

Definition 6 *A Nisnevich cover $U \to X$ is an étale cover such that for any field K, the induced map on K-points is surjective.*

Tensor Products in PST:
$\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) := \mathbb{Z}_{tr}(X \times Y)$.
$\mathbb{Z}_{tr}(X)$ projective in **PST**; $\bigoplus_\alpha \mathbb{Z}_{tr}(X_\alpha) \twoheadrightarrow F$
$R_F \to F,\ R_G \to G$ resolutions by representable sheaves.
$F \otimes G := Tot(R_F \otimes R_G) = F \otimes^{\mathbf{L}} G$.

$\mathbb{A}^1$-weak equivalence:

$$\mathbf{D}^- := \mathbf{D}^-(\mathbf{Sh_{Nis}}(\mathbf{Cor_k}))$$

Quotient $\mathbf{D}^-$ by smallest thick subcategory W containing cones of $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$.

Definition 7
$\mathbf{DM^{eff}_{Nis}(k)} := \mathbf{D^-Sh_{Nis}(Cor_k)[W^{-1}]}$.

Motives (Continued)

Definition 8 *$M(X) := \mathbb{Z}_{tr}(X) \in \mathbf{DM}^{\mathbf{eff}}_{\mathbf{Nis}}(\mathbf{k})$. $\mathbf{DM}^{\mathbf{eff}}_{\mathbf{geo}}(\mathbf{k}) \subset \mathbf{DM}^{\mathbf{eff}}_{\mathbf{Nis}}(\mathbf{k})$ is the thick subcategory generated by the $M(X)$.*

Why the Nisnevich topology?

Proposition 9 *$U \to X$ Nisnevich cover. Then*

$$\cdots \to \mathbb{Z}_{tr}(U \times_X U) \to \mathbb{Z}_{tr}(U) \to \mathbb{Z}(X) \to 0$$

exact sequence of Nisnevich sheaves.

Suffices to check for $X = \operatorname{Spec} A$, A henselian local. Now use the fact that a Nisnevich cover of $\operatorname{Spec} A$ is split.

Properties of $\mathbf{DM}^{\mathbf{eff}}_{\mathbf{Nis}}(\mathbf{k})$

Mayer-Vietoris:
$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$.

Kunneth: $M(X \times Y) \cong M(X) \otimes M(Y)$.

Vector Bundle: $M(X) \cong M(V)$ when $V \to X$ vector bundle.

Cancellation: Define $M(q) = M \otimes \mathbb{Z}(q)$. Assume varieties over k admit resolution of singularities. Then $Hom(M, N) \cong Hom(M(q), N(q))$.

Projective Bundle:
$\bigoplus_{i=0}^{n} M(X)(i)[2i] \cong M(\mathbb{P}(V))$.

Chow Motives: X, Y smooth projective. Then $Hom(M(X), M(Y)) \cong CH^{\dim X}(X \times Y)$.

Relation with Motivic Cohomology

Theorem 10 *X/k smooth. Then*
$H^{p,q}(X) \cong Hom_{\mathbf{DM}^{\mathbf{eff}}_{\mathbf{Nis}}(\mathbf{k})}(\mathbb{Z}_{tr}(X), \mathbb{Z}_{tr}(q)[p])$.

Algebraic Cycles

$\Delta^\bullet$ cosimplicial variety. $\mathcal{Z}^q(X,n) =$ codim. q algebraic cycles on $X \times \Delta^n$ in good position for faces. $\mathcal{Z}^q(X,\bullet)$:
$\cdots \to \mathcal{Z}^q(X,2) \to \mathcal{Z}^q(X,1) \to \mathcal{Z}^q(X,0).$

$$CH^q(X,n) := H^{-n}(\mathcal{Z}^q(X,\bullet)).$$

Theorem 11 (Voevodsky) *X smooth, k perfect. Then*

$$H^{p,q}(X) \cong CH^q(X, 2q-p).$$

Better to work with $\mathcal{Z}^q(X,\bullet)[-2q]$.

$$H^{p,q}(X) = H^p(\mathcal{Z}^q(X,\bullet)[-2q]).$$

Beilinson, Soulé conjecture that this shifted complex has cohomological support in degrees $[0, 2q]$.

Relation with K-Theory

$$CH^q(X,n)_{\mathbb{Q}} \cong gr_{\gamma}^q K_n'(X)_{\mathbb{Q}}.$$

Relation with Étale Cohomology

Theorem 12 (Suslin) *Let $k = \bar{k}$ and let X/k be smooth. Let $m \geq 1$ be relatively prime to char. k. Then*

$$H^p_{et}(X, \mathbb{Z}/m\mathbb{Z}(q)) \cong H^p\Big(\mathcal{Z}^q(X,\bullet)[-2q] \otimes \mathbb{Z}/m\mathbb{Z}\Big).$$

Arithmetic Conjectures

Conjecture 13 (Soulé) *Let X be an arithmetic variety of dimension n. Define the scheme-theoretic zeta function*

$$\zeta_X(s) := \prod_{\substack{x \in X \\ x\, clsd.}} \frac{1}{1 - N(x)^{-s}}.$$

Then the Euler-Poincaré characteristic of the complex $\chi(X,q)$ of $\mathcal{Z}^q(X,\bullet)[-2q]$ is finite, and we have $\chi(X,q) = -ord_{s=n-q}\zeta_X(s)$.

Example 14 *$X = \mathcal{O}_k$, ring of integers in a number field, $n = 1$. By Borel*

$$rkH^{p,q}(\mathcal{O}_k) = \begin{cases} 1 & p = q = 0 \\ r_1 + r_2 - 1 & q = p = 1 \\ r_1 + r_2 & p = 1;\ q = 2m+1 \\ r_2 & p = 1;\ q = 2m \\ 0 & else. \end{cases}$$

Arithmetic Conjectures (Conjectures of Beilinson; Bloch, Kato; Fontaine, Perrin-Riou)

Regulator map: $k \subset \mathbb{C}$, X smooth projective. $r : H^{p,q}(X) \to H^{p,q}_{\mathcal{D}}(X)$. Assume $p < 2q$ (negative weight). Then

$$0 \to H^{p-1}(X,\mathbb{C})/(F^q + H^{p-1}(X,\mathbb{Z}(q)) \to H^{p,q}_{\mathcal{D}}(X) \to H^p(X,\mathbb{Z}(q))_{tors} \to 0.$$

Must also take invariants under real conjugation F_∞: $H^{p,q}_{\mathcal{D}}(X)_{\mathbb{R}}$

Volume form given (upto $\mathbb{Q}^\times$) by rational structure on H^p_{DR}/F^q.

Take $p < 2q-1$ for simplicity. Let $L(H^{p-1}(X,\mathbb{Q}_\ell), s)$ be the Hasse-Weil L-function. Let $L^*(s = p-q)$ be the first non-vanishing term in the Taylor series at $s = p-q$.

Arithmetic Conjectures (cont.)

Conjecture 15 *(i) Image of* r *is cocompact in* $H^{p,q}_{\mathcal{D}}(X)_{\mathbb{R}}$.
(ii) Rank of $H^{p,q}(X) \otimes \mathbb{Q}$ *is the order of vanishing of* $L(H^{p-1}(X, \mathbb{Q}_\ell), s)$ *at* $s = p - q$.
(iii) Volume of $H^{p,q}_{\mathcal{D}}(X)_{\mathbb{R}}/\mathit{Image}(r) \in L^*(H^{p-1}(X, \mathbb{Q}_\ell(q)), s = p - q) \cdot \mathbb{Q}^\times$.

Fontaine, Perrin-Riou: Reformulate in terms of (roughly) a metric on $\det H^{p,q}(X)$. (The metric depends on the trivialization of a certain Betti-DR line.)

Problem: Construct this metric directly on $\mathcal{Z}^q(X, \bullet)[-2q]$.

Mixed Tate Motives

k field. M motive over k. (Conjectural) t-structure on $DM(k)$, $M \in$ core. Again conjecturally, M will have a weight filtration $W_\bullet M$.

Definition 16 *M is mixed Tate if* $gr^W M = \bigoplus_i \mathbb{Z}(n_i)$.

Problem: Give a *synthetic* construction of mixed Tate motives over k.

1. (Deligne, Goncharov,...) Look at subquotients of the groupring $\mathbb{Z}[\pi_1(\mathbb{P}^1_k - S, p_0)]$.

2. (Bloch, Kriz) (a) Construct a DG version $\bigoplus_{q \geq 0} \mathfrak{N}^*(q)$ of $\bigoplus_q \mathcal{Z}^q(\operatorname{Spec} k, \bullet)[-2q] \otimes \mathbb{Q}$.
(b) Consider corepresentations of the the commutative Hopf algebra

$$H = H^0(Bar(\bigoplus_q \mathfrak{N}^*(q))).$$

$I \subset H$ augmentation ideal. Conjecturally, $\mathcal{L} := (I/I^2)^\vee$ is the pro-Lie algebra associated to the Tannaka group of mixed Tate motives/k.

Limiting Mixed Motives

(Important work on this subject by Ayoub, Bondarko,Vologodsky,... What follows is a modest attempt to interpret LMM in terms of cycle complexes.)

X/k smooth, $Y = \bigcup_{i=1}^{r} Y_i \subset X$ normal crossings divisor, $X^* = X - Y$. Write $\mathfrak{N}(X) = \mathfrak{N}^*(X, **)$ (i.e. ignore gradings)

$$0 \to \mathfrak{N}(Y) \to \mathfrak{N}(X) \to \mathfrak{N}(X^*) \to \mathcal{C} \to 0$$

Here $\mathcal{C} \simeq 0$.

Weight filtration:
$Y_I := \bigcup_{i\in I} Y_i,\ Y^I := \bigcup_{j\notin I} Y_j$.
$\mathfrak{N}(X, Y_I) := \mathfrak{N}(X)/\mathfrak{N}(Y_I)$
$\mathfrak{N}(X)_{Y^I} \subset \mathfrak{N}(X)$ cycles meeting Y^I properly (including faces).
$\mathfrak{N}(X, Y_I)_{Y^I} = \mathfrak{N}(X)_{Y^I}/\mathfrak{N}(X)_{Y^I} \cap \mathfrak{N}(Y_I)$
$W_p\mathfrak{N}(X, Y) := \sum_{|I|=p} \mathfrak{N}(X, Y_I)_{Y^I} \subset \mathfrak{N}(X, Y)$.
$W_0 = \mathfrak{N}(X)_Y \simeq \mathfrak{N}(X)$.
$W_r = \mathfrak{N}(X, Y) \simeq \mathfrak{N}(X^*)$.

Limiting Mixed Motives (cont.)

Theorem 17 $Y(I) := \bigcap_{i\in I} Y_i,\ Y(\emptyset) := X$. *Then*

$$gr_p^W \mathfrak{N}(X,Y;q) \simeq \oplus_{|I|=p} \mathfrak{N}(Y(I);q-p)[-p].$$

Analogy: $W_\bullet \mathfrak{N}(X,Y) \leftrightarrow W_\bullet \Omega_X^*(\log Y)$.

Now assume $Y : t = 0$ principal.

Steenbrink double complex:

$A^{pq} := \Omega_X^{p+q+1}(\log Y)/W_q$.

$d' = d : A^{p,q} \to A^{p+1,q}$;

$d'' = \wedge dt/t : A^{p,q} \to A^{p,q+1}$

Motivic analog:

$\mathfrak{N} = \mathfrak{N}(X,Y),\ \mathfrak{M} = \bigoplus_{i=0}^{r-1} \mathfrak{N}/W_i$.

$W_a\mathfrak{M} := \bigoplus_{b=0}^{r-1} W_{2b+a+1}\mathfrak{N}/W_b\mathfrak{N}$.

$gr_a^W \mathfrak{M} \simeq$

$\bigoplus_{b=0}^{r-1} \bigoplus_{|I|=2b+a+1} H^{*,*}(Y(I))[-2b-a-1]$.

Motivic analog (cont.)

$A^{**} \leftrightarrow \mathfrak{M}$. Construct
$B : \mathfrak{M} \to \mathfrak{M},\ B^2 = 0,\ B(W_a) \subset W_a$. Then
$B \leftrightarrow d' + d''$ in Steenbrink.

Definition 18 *The limiting mixed motivic cohomology is $H^{*,*}(\mathfrak{M}, B)$.*

Corollary 19 *"Steenbrink spectral sequence":*

$$E_1 = \bigoplus_{b=0}^{r-1} \bigoplus_{|I|=2b+a+1} H^{*,*}(Y(I))[-2b-a-1] \Rightarrow H^{*,*}(\mathfrak{M}, B)$$

Warning: B depends on a choice of homotopy. I have not yet checked the dependence of the complex $(\mathfrak{M}, B)$ on this choice.