Thoughts on graph polynomials and related questions

(A pot-pourri of partial results)

I. Quillen metrics for dummies

II. Singularities of graph hypersurfaces

A. Dual hypersurface interpretation; the incidence correspondence

B. Transversality; Patterson's theorem

C. Stratified Morse functions

III. External momenta

A. Second Symanzik polynomial

B. One loop graphs.

Quillen metrics for dummies (with Carly Klivans)

Polynomials for metrized CW-complexes generalizing the Kirchhoff polynomial for metrized graphs.

$K^\bullet = \bigoplus K^i$ finite dim. graded $\mathbb{R}$-vector space.

$K^i, \langle \bullet, \bullet \rangle$ symm. positive definite inner products

$$d : K^i \to K^{i+1},\ d^2 = 0;\ \langle dx, y \rangle = \langle x, d^* y \rangle$$

$$\Delta^i = dd^* + d^* d; K^i \to K^i;\ \Delta^\bullet = \oplus \Delta^i$$

$$B^\bullet = \mathrm{Image}(d);\ C^\bullet = \mathrm{Image}(d^*);$$

$$K^\bullet = \ker \Delta^\bullet \oplus B^\bullet \oplus C^\bullet.$$

Quillen metrics for dummies (bis)

Determinant line:

$$\det(K^\bullet) = \bigotimes_i (\det H^i)^{(-1)^i} \overset{(*)}{=} \bigotimes_i (\det K^i)^{(-1)^i}.$$

$\det H^i$, $\det K^i$ have metrics.

$(*)$ canonical, but not an isometry.

$$\det(K^\bullet)_{L^2}; \quad \det(K^\bullet)_Q$$

Proposition 1 $||x||^2_{L^2}/||x||^2_Q = \det(\Delta^\bullet|C^\bullet)$.

Proposition 2

$$||x||^2_Q/||x||^2_{L^2} = \prod_{i=0}^{n} (\det_{\neq 0} \Delta^i)^{i(-1)^i}.$$

Quillen metrics for dummies (Special Case)

Suppose $K^\bullet = K^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$; $d : K^i_{\mathbb{Z}} \to K^{i+1}_{\mathbb{Z}}$.

$$K^\bullet_{\mathbb{Z}} = K^{ev}_{\mathbb{Z}} \oplus K^{odd}_{\mathbb{Z}};$$

basis $\{e^x_j\}$, $x = ev, odd$.

$$\langle e^x_i, e^x_j \rangle = a^x_i \delta_{ij}, \; x = ev, odd.$$

$$T, S' \subset \{e^{ev}_j\}; \quad T', S \subset \{e^{odd}_j\}$$

$$\widetilde{S} = \{e^{odd}_j\} - S, \; \widetilde{T} = \{e^{ev}_j\} - T, \ldots$$

Image and kernel of d

$$B^x_{\mathbb{Z}} \subset Z^x_{\mathbb{Z}} \subset K^x_{\mathbb{Z}}; \; x = ev, odd.$$

$$\mathbb{Z}T \oplus Z^{ev}_Z \xrightarrow{\text{isog.}} K^{ev}_{\mathbb{Z}}; \quad \mathbb{Z}\widetilde{S} \oplus B^{odd}_Z \xrightarrow{\text{isog.}} K^{odd}_{\mathbb{Z}}$$

$$\mathbb{Z}T' \oplus Z^{odd}_Z \xrightarrow{\text{isog.}} K^{odd}_{\mathbb{Z}}; \quad \mathbb{Z}\widetilde{S}' \oplus B^{ev}_Z \xrightarrow{\text{isog.}} K^{ev}_{\mathbb{Z}}$$

Quillen metrics for dummies (Special Case (bis))

Define

$$|T| = \#(B_{\mathbb{Z}}^{odd}/d\mathbb{Z}T); \quad |S| = \#(K_{\mathbb{Z}}^{odd}/(B_{\mathbb{Z}}^{odd} \oplus \widetilde{S})$$

and similarly for S', T'.

Theorem 3

$$||x||_Q^2/||x||_{L^2} =$$

$$\frac{\Big(\sum_{T'}(\prod_{j\in T'} a_j^{odd})^{-1}|T'|^2\Big)\Big(\sum_{S'}(\prod_{j\in S'} a_j^{ev})|S'|^2\Big)}{\Big(\sum_{T}(\prod_{j\in T} a_j^{ev})^{-1}|T|^2\Big)\Big(\sum_{S}(\prod_{j\in S} a_j^{odd})|S|^2\Big)}$$

Take $x = (\bigwedge e_i^{ev}) \otimes (\bigwedge e_i^{odd})^{-1}$. Then

$$||x||_{L^2} =$$

$$\frac{\Big(\sum_{T}(\prod_{j\in T'} a_j^{ev})|T|^2\Big)\Big(\sum_{S}(\prod_{j\in S} a_j^{odd})|S|^2\Big)}{\Big(\sum_{T'}(\prod_{j\in T'} a_j^{odd})|T'|^2\Big)\Big(\sum_{S'}(\prod_{j\in S'} a_j^{ev})|S'|^2\Big)}$$

Quillen metrics for dummies (graphs)

G connected graph; vertices v_i, edges e_i.

$$\langle e_i, e_j \rangle = \delta_{ij}; \quad \langle v_i, v_j \rangle = a_i \delta_{ij}$$

x as above

$$||x||_{L^2} = \frac{1}{\left(\sum_{t=\text{span. tree}} \prod_{e_i \notin t} a_i \right) (\#\text{vertices})}$$
$$= \frac{1}{(\#\text{vertices}) \cdot \text{Kirchhoff polynomial of } G}$$

Question: Is there some physical interpretation of this rational function for higher dimensional CW-complexes analogous to electrical flow through a graph.

Graph hypersurfaces as dual hypersurfaces (+results of Eric Patterson)

G a graph. E edges of G. $H = H_1(G, \mathbb{Q})$. $e \in E$

$$e^\vee : H \to \mathbb{Q}; \quad \ell = \sum n_e e \mapsto n_e.$$

Write $P = \mathbb{P}(H) := Proj(Sym(H^\vee))$:

$$\mathcal{L} = \sum_E \mathbb{Q} \cdot e^{\vee,2} \subset \Gamma(P, \mathcal{O}(2)).$$

Assume the $e^{\vee,2}$ linearly independent, $n = \#E$.

$$|\mathcal{L}| : P \to \mathbb{P}^{n-1}$$

Finite map, everywhere defined. Not an embedding (usually).

Graph hypersurface is the dual hypersurface

$$X \subset \mathbb{P}^{n-1,\vee}$$

$x \in X \leftrightarrow Q_x \subset P$ singular quadric.

Graph hypersurfaces as dual hypersurfaces (Duality)

$$\Lambda = \{(x,y)\ |x \in X,\ y \in Q_{x,sing} \subset \mathbb{P}(H)\} \subset X \times \mathbb{P}(H)$$

$$\begin{array}{ccc} & \Lambda & \\ \swarrow p & & q \searrow \\ X & & \mathbb{P}(H) \end{array}$$

p birational, fibres projective spaces. Unlike classical case, Λ may have singularities.

$$\Lambda_{sing} \leftrightarrow G = G_1 \cup G_2;$$

$$h_1(G_i) > 0,\ \text{no common edges.}$$

Theorem 4 $H^*(\Lambda, \mathbb{Z})$ *mixed Tate, uninteresting.*

Problem: Use this picture to understand the topology of X.

The rank stratification

Stratify X by $\dim p^{-1}(x)$

$$X = \coprod_{i \geq 0} X_i; \quad X_i = \{x \mid \dim p^{-1}(x) = i\}$$

$X : \Psi_G = 0$ graph polynomial.

Theorem 5 *(Patterson) $x \in X$. Then the multiplicity of Ψ_G at x equals $1 + \dim p^{-1}(x)$.*

Corollary 6 *The smooth locus of X is the locus where p is an isomorphism.*

The Universal Case

$r = \dim H$,
$\mathbb{P}^{r(r+1)/2-1} = \mathbb{P}(\Gamma(\mathbb{P}(H), \mathcal{O}(2))) =$
universal family of $r \times r$ symmetric matrices.
$\mathcal{X} \hookrightarrow \mathbb{P}^{r(r+1)/2-1}$ hypersurface defined by universal determinant.

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{P}^{n-1,\vee} & \xrightarrow{\iota} & \mathbb{P}^{r(r+1)/2-1} \end{array}$$

Diagram of embeddings.
Fix basis of H. $e^{\vee,2}$ rk 1 symmetric matrix.
Image of ι equals span of the $e^{\vee,2}$. Patterson's thm says X and $\mathcal{X}$ have the same multiplicity at $x \in X$. It is not true, however, that ι is transverse to the higher rank strata of $\mathcal{X}$.

Example

$x \in X \leftrightarrow \sum_e x_e e^{\vee,2}$ symmetric matrix with $V = \ker x \subset H$. Assume $\dim V = 2$, i.e. $x \in X_1$.

Choose e_1, e_2 so that $e_1^\vee \oplus e_2^\vee : V \cong \mathbb{Q}^2$.
$0 \neq v_i \in \ker e_i^\vee$.

Then ι transverse to $\mathcal{X}_1$ iff $\{e^{\vee,2}|_V\}$ span 3 dim. space of quad. forms on V.

v_1, v_2 loops on G with no common edges.
Every $e^{\vee,2}|_V$ zero or proportional to exactly one of $e_i^{\vee,2}|_V$.

Must find $\sum x_e e^{\vee,2}$ with exactly V as null space. Two linear conditions on $\{x_e\}$.

Stratified Morse Theory

Universal case, $\mathcal{P} := \mathbb{P}_{\mathbb{C}}^{r(r+1)/2-1}$.

$$\mathcal{X}_{\mathbb{C}} \subset \mathcal{P} \xrightarrow{f} \mathbb{R}$$

Stratification $\mathcal{P} - \mathcal{X}_{\mathbb{C}},\ \mathcal{X}_0, \mathcal{X}_1, \ldots\ p \in \mathcal{P}$.

$$df_p \in T^*_{\mathcal{P},p,\mathbb{R}} = T^*_{\mathcal{P},p} \oplus \overline{T}^*_{\mathcal{P},p}$$

Assume $p \in \mathcal{X}_i$

$$0 \to N^*_{\mathcal{X}_i \subset \mathcal{P},p,\mathbb{R}} \to T^*_{\mathcal{P},p,\mathbb{R}} \to T^*_{\mathcal{X}_i,p,\mathbb{R}} \to 0$$

Stratified critical point: $df_p \in N^*_{\mathcal{X}_i \subset \mathcal{P},p,\mathbb{R}}$

Problem: Construct a stratified Morse function for $X \hookrightarrow \mathbb{P}^{n-1,\vee}$ by restricting a Morse function from $\mathcal{X} \hookrightarrow \mathcal{P}$. Use it to get information about $H_*(X)$.

A First Step

$z_{ij}, 1 \leq i \leq j \leq r$ homogeneous coordinates on $\mathcal{P}$. Define

$$f := \sum c_i c_j \rho_{ij} |z_{ij}|^2 / \sum \rho_{ij} |z_{ij}|^2$$

$$\rho_{ij} = \begin{cases} 2 & i < j \\ 1 & i = j \end{cases}; \quad c_i > 0, \text{ generic}$$

Computation: f has isolated critical points at symmetric matrices 1_{ii} and $1_{ij} + 1_{ji}$; no other critical points on strata. Technically, f is not a stratified Morse function, but it is an interesting first step.

Problems: Many. How to understand the restriction to $X \subset \mathbb{P}^{n-1,\vee}$? How to understand the "links" for the strata.

External Momenta (with Dirk Kreimer)

Graph polynomials as functions of external momenta (aka 2nd Symanzik polynomial)

G connected graph

$$\begin{array}{ccccccccc} 0 & \to & H_1(G,\mathbb{R}) & \to & \mathbb{R}^E & \xrightarrow{\partial} & \mathbb{R}^{V,0} & \to & 0 \\ & & \| & & \uparrow & & \uparrow q & & \\ 0 & \to & H_1(G,\mathbb{R}) & \to & V_q & \to & \mathbb{R} & \to & 0 \end{array}$$

$V_q = \partial^{-1}(\mathbb{R}q) \subset \mathbb{R}^E$.

Theorem 7 *(Patterson) "Configuration polynomial" for $V_q \subset \mathbb{R}^E$ is the second Symanzik polynomial for G with (scalar) external momentum q.*

Basis h_i for $H_1(G)$; $h_q \in V_q$ lifting $q(1) \in \mathbb{R}^{V,0}$.
$e \in E,\ e^\vee : V_q \to \mathbb{R}$.
$w_e = (e^\vee(h_1), e^\vee(h_2), \ldots, e^\vee(h_q))$.

$$\Psi^{(2)}_{G,q}(A) = \det(\sum_{e\in E} A_e w_e^t w_e).$$

4-vector external momenta

$\mathcal{A} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ quaternions. Conjugation $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$. Do construction from previous slide in $\mathcal{A}$.

$$\begin{array}{ccccccccc} 0 & \to H_1(G) \otimes \mathcal{A} & \to \mathcal{A}^E & \xrightarrow{\partial} & \mathcal{A}^{V,0} & \to 0 \\ & \| & \uparrow & & \uparrow q & \\ 0 & \to H_1(G) \otimes \mathcal{A} & \to V_q & \to & \mathcal{A} & \to 0 \end{array}$$

Now take $x_e := \bar{w}_e^t \cdot w_e$. Have $\sum_e A_e x_e = r \times r$-quaternionic hermitian matrix.

Conjecture 8 $\Psi^{(2)}_{G,q}(A) = Nrp(\sum A_e x_e)$. *$Nrp^2 = Nrd$ quaternionic Pfaffian (square root of reduced norm) (E.H. Moore).*

Example: G one loop

G one loop, $H_1 = \mathbb{R}(e_1 + \cdots + e_n)$. $\sum \mu_e e \in \mathcal{A}^E$ lifting $q(1) \in \mathcal{A}^{V,0}$.

$$\sum A_e x_e = \begin{pmatrix} \sum_e A_e & \sum_e A_e \mu_e \\ \sum A_e \bar{\mu}_e & \sum A_e \bar{\mu}_e \mu_e \end{pmatrix} \tag{1}$$

$$\Psi^{(2)}_{G,q}(A) = \sum_{i<j} \overline{(q_i + \cdots + q_{j-1})}(q_i + \cdots + q_{j-1}) A_i A_j .$$

$$Q_{G,q,m}(A) = (\sum A_i)(\sum m_i^2 A_i) + \Psi^{(2)}_{G,q}(A)$$

Feynman Amplitudes
One loop with external momenta

(work of Davydychev and Delbourgo) Feynman period, 6 edges:
($\Omega_5 = \sum \pm A_i dA_1 \wedge \ldots \widehat{dA_i} \ldots \wedge dA_6$.)

$$\int_\sigma \frac{(\sum A_i)^2 \Omega_5}{Q_{G,q,m}(A)^4}; \quad \sigma = \{A_i \geq 0\} \tag{2}$$

Goncharov construction of mixed Tate motives: Z smooth quadric in good position.

$H = H^{2n+1}(\mathbb{P}^{2n+1} - Z, \Delta - \Delta \cap Z); \quad \Delta : \prod A_i = 0.$

Case $n = 2$

$$gr^W H = \mathbb{Q} \oplus \mathbb{Q}(-1)^{15} \oplus \mathbb{Q}(-2)^{15} \oplus \mathbb{Q}(-3)$$

Extraordinary fact: Feynman integrand (2) sits in $W_4 H \subsetneq W_6 H = H$.
D+D interpretation: Feynman amplitude = sum of dilogarithms.

Concretely

$$d(\sum_i \frac{L_i \Xi_i}{Q^3}) \stackrel{?}{=} \frac{(\sum A_i)^2 \Omega_5}{Q_{G,q,m}(A)^4}$$

Ξ_i 4-forms analogous to Ω_5.

So what? Outstanding problems:

a. Gauß-Manin differential equation in q.

b. Monodromy about Landau singularities

Remark 9 *Restriction to* 6 *edges probably not important. Recall* 2×2 *quaternionic hermitian matrix:*

$$\begin{pmatrix} \sum_e A_e & \sum_e A_e \mu_e \\ \sum A_e \bar{\mu}_e & \sum A_e \bar{\mu}_e \mu_e \end{pmatrix}$$

Space of such has $\dim_{\mathbb{R}} = 6$. *Family parametrized by* $\mathbb{P}^5$ *is universal.*