A NOTE ON TWISTOR INTEGRALS

SPENCER BLOCH

1. Introduction

This paper is a brief introduction to twistor integrals from a mathematical point of view. It was inspired by a paper of Hodges [H] which we studied in a seminar at Cal Tech directed by Matilde Marcoli. The idea is to write the amplitude for a graph with n loops and $2n+2$ propagators using the geometry of pfaffians for sums of rank 2 alternating matrices. (Hodges considers the case of 1 loop and 4 edges). Why is this of interest to a mathematician? The Feynman amplitude is a *period* in the sense of arithmetic algebraic geometry. In parametric form, the amplitude integral associated to a graph Γ with N edges and n loops has the form

$$c(N,n)\int_\delta \frac{S_1^{N-2n-2}\Omega}{S_2^{N-2n}}. \tag{1.1}$$

Here S_1 and S_2 are the first and second Symanzik polynomials [BK], [BEK], [IZ], and $\Omega = \sum \pm A_i dA_1 \wedge \cdots \wedge \widehat{dA_i} \wedge \cdots \wedge dA_N$ is the integration form on $\mathbb{P}^{N-1}$, the projective space with homogeneous coordinates indexed by edges of Γ. The chain of integration δ is the locus of points on $\mathbb{P}^{N-1}$ where all the $A_i \geq 0$. Note $\Omega,\ S_1,\ S_2$ are homogeneous of degrees $N,\ n,\ n+1$ in the A_i, so the integrand is homogeneous of degree 0 and represents a rational differential form. Finally, $c(N,n)$ is some elementary constant depending only on N and n.

Two special cases suggest themselves. In the *log divergent* case when $N = 2n$, the integrand is simply Ω/S_1^2. The first Symanzik polynomial depends only on the edge variables A_i, so the result in this case is a constant. (If the graph is non-primitive, i.e. has log divergent subgraphs, the integral will diverge. We do not discuss this case.) Inspired by the conjectures of Broadhurst and Kreimer [BrK], there has been a great deal of work done on the primitive log divergent amplitudes.

The polynomial S_1 itself is the determinant of an $n \times n$-symmetric matrix with entries linear forms in the A_i. The linear geometry of this determinant throws an interesting light on the motive of the hypersurface $X(\Gamma) : S_1 = 0$. For example, one has a "Riemann-Kempf" style theorem that the dimension of the null space of the matrix at a point is equal to the multiplicity of the point on $X(\Gamma)$, [P], [K]. Furthermore, the projectivized fibre space $Y(\Gamma)$ of these null lines maps birationally onto $X(\Gamma)$ and in some sense "resolves" the motive. Whereas the motive of $X(\Gamma)$ can be quite subtle, the motive of $Y(\Gamma)$ is quite elementary. In particular, it is mixed Tate [B]. (The Riemann-Kempf theorem refers to the map $\pi : Sym^{g-1}C \twoheadrightarrow \Theta \subset J_{g-1}(C)$ where C is a Riemann surface and Θ is the theta divisor. The dimension of the fibre of π at a point of Θ equals the multiplicity of the divisor Θ at the point minus one.)

The second case is $N = 2n + 2$, e.g. one loop and 4 edges. The amplitude is $\int_\delta \Omega/S_2^2$ and is a function of external momenta and masses. The second Symanzik has the form

$$S_2 = S_2^0(A, q) - (\sum_{i=1}^{N} m_i^2 A_i) S_1(A) \tag{1.2}$$

Here q denotes the external momenta, and $S_2^0(A, q)$ is homogeneous of degree 2 in q and of degree $n+1$ in the A. Moreover, S_2^0 is a quaternionic pfaffian associated to a quaternionic hermitian matrix, [BK], so in the case of zero masses there is again the possibility of linking the motive to the geometry of a linear map. In this note we go further and show for the case $N = 2n + 2$ that S_2 is itself a pfaffian via the calculus of *twistors*.

To avoid issues with convergence for the usual propagator integral, I assume in what follows that the masses are positive and the propagators are euclidean. Note that in (1.4) the pfaffian can vanish where some of the $a_i = 0$. The issues which arise are analogous to issues of divergence already familiar to physicists. They will not be discussed here.

Theorem 1.1. *Let Γ be a graph with n loops and $2n + 2$ edges as above. We fix masses $m_i > 0$ and external momenta q and consider the amplitude*

$$\mathcal{A}(\Gamma, q, m) = \int_{\mathbb{R}^{4n}} \frac{d^{4n}x}{\prod_{i=1}^{2n+2} P_i(x, q, m_i)} \tag{1.3}$$

where the P_i are euclidean. Then there exist alternating bilinear forms Q_i on $\mathbb{R}^{2n+2}$ where Q_i depends on P_i, $1 \le i \le 2n + 2$, and a universal constant $C(n)$ depending only on n such that

$$\mathcal{A}(\Gamma, q, m) = C(n) \int_\delta \frac{\Omega_{2n+1}}{\mathit{Pfaffian}(\sum_{i=1}^{2n+2} a_i Q_i)^2} \tag{1.4}$$

Here $\Omega_{2n+1} = \sum \pm a_i da_1 \wedge \cdots \widehat{da_i} \cdots da_{2n+2}$ *and* δ *is the locus on* $\mathbb{P}^{2n+1}$ *with coordinate functions* a_i *where all the* $a_i \geq 0$.

By way of analogy, the first Symanzik polynomial is given by

$$S_1(\Gamma)(a_1, \ldots, a_N) = \det(\sum_{e \text{ edge}} a_e M_e) \tag{1.5}$$

where M_e is a rank 1 symmetric $n \times n$-matrix associated to $(e^\vee)^2$, where $e^\vee : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$ is the functional which associates to a loop the coefficient of e in that loop. Thus, the amplitude in the case of n loops and $2n$ edges is given by

$$\mathcal{A}(\Gamma) = C'(n) \int_\delta \frac{\Omega_{2n-1}}{\det(\sum a_i M_i)^2} \tag{1.6}$$

where $C'(n)$ is another constant depending only on n.

I want to acknowledge help from S. Agarwala, M. Marcolli, and O. Ceyhan. Much of this work was done during June, 2012 when I was visitng Cal Tech.

2. Linear Algebra

Fix $n \geq 1$ and consider a vector space $V = k^{2n+2} = ke_1 \oplus \cdots \oplus ke_{2n+2}$. (Here k is a field of characteristic 0.) We write $O = ke_1 \oplus ke_2$ and $I = ke_3 \oplus \cdots \oplus ke_{2n+2}$, so $V = O \oplus I$. $G(2, V)$ will be the Grassmann of 2-planes in V.

We have

$$\mathrm{Hom}_k(O, I) \overset{\iota}{\hookrightarrow} G(2, V) \overset{j}{\hookrightarrow} \mathbb{P}(\bigwedge^2 V). \tag{2.1}$$

Here $\iota(\psi) = k(e_1 + \psi(e_1)) \oplus k(e_2 + \psi(e_2))$ and $j(W) = \bigwedge^2 W \hookrightarrow \bigwedge^2 V$.

Write V^* for the dual vector space with dual basis e_i^*. We identify $\bigwedge^2 V^*$ with the dual of $\bigwedge^2 V$ in the evident way, so $\langle e_i^* \wedge e_j^*, e_i \wedge e_j \rangle = 1$. For $\alpha \in \bigwedge^2 V^*$, the assignment

$$\psi \mapsto \langle (e_1 + \psi(e_1)) \wedge (e_2 + \psi(e_2)), a \rangle \tag{2.2}$$

defines a quadratic map $q_\alpha : \mathrm{Hom}(O, I) \to k$.

Lemma 2.1. *Assume* $0 \neq \alpha = v \wedge w$ *with* $v, w \in V^*$. *Then the quadratic map* q_α *has rank* 4.

Proof. It suffices to show $\langle (\sum x_i e_i) \wedge (\sum y_j e_j), v \wedge w \rangle$, viewed as a quadric in the x_i and y_j variables, has rank 4. By assumption v, w are linearly independent. We can change coordinates so $v = \varepsilon_i^*, w = \varepsilon_j^*$, and $\sum x_i e_i = \sum {x'}_i \varepsilon_i, \sum y_j e_j = \sum {y'}_j \varepsilon_j$. The polynomial is then

$$\langle (\sum {x'}_i \varepsilon_i) \wedge (\sum {y'}_j \varepsilon_j), \varepsilon_i^* \wedge \varepsilon_j^* \rangle = {x'}_i {y'}_j - {x'}_j {y'}_i. \tag{2.3}$$

This is a quadratic form of rank 4. □

Returning to the notation in (2.1), we can write $I = \bigoplus_{i=1}^n I_i$ with $I_i = ke_{2i+1} \oplus ke_{2i+2}$. We can think of $\mathrm{Hom}(O, I) = \bigoplus \mathrm{Hom}(O, I_i)$ as the decomposition of momentum space into a direct sum of Minkowski spaces. We identify $\mathrm{Hom}(O, I_i)$ with the space of 2×2-matrices, and the propagator with the determinant. With these coordinates, an element in $\mathrm{Hom}(O, I)$ can be written as a direct sum $A_1 \oplus \cdots \oplus A_n$ of 2×2-matrices. The propagators have the form $\det(a_1 A_1 + \cdots a_n A_n)$ with $a_i \in k$. The map $\psi : O \to I$ given by $\psi(e_1) = x_3 e_3 + \cdots + x_{2n+2} e_{2n+2}$ and $\psi(e_2) = y_3 e_3 + \cdots + y_{2n+2} e_{2n+2}$ corresponds to the matrices

$$A_i = \begin{pmatrix} x_{2i+1} & x_{2i+2} \\ y_{2i+1} & y_{2i+2} \end{pmatrix}. \tag{2.4}$$

Lemma 2.2. *Let A_i be as in (2.4). Let*

$$\alpha = (\sum_{i=1}^n a_i e_{2i+1}^*) \wedge (\sum_{i=1}^n a_i e_{2i+2}^*) \in \bigwedge^2 V^*.$$

Then the quadratic map q_α in lemma 2.1 is given by

$$q_\alpha(A_1 \oplus \cdots \oplus A_n) = \det(a_1 A_1 + \cdots + a_n A_n). \tag{2.5}$$

Proof. This amounts to the identity

$$\det \begin{pmatrix} \sum a_i x_{2i+1} & \sum a_i x_{2i+2} \\ \sum a_i y_{2i+1} & \sum a_i y_{2i+2} \end{pmatrix} = \tag{2.6}$$
$$\langle (\sum_{i\geq 3} x_i e_i) \wedge (\sum_{i \geq 3} y_i e_i), (\sum_{i=1}^n a_i e_{2i+1}^*) \wedge (\sum_{i=1}^n a_i e_{2i+2}^*) \rangle.$$

For $i = j$ (resp. $i \neq j$) the coefficient of $a_i a_j$ in this expression is

$$x_{2i+1} y_{2i+2} - x_{2i+2} y_{2i+1} \tag{2.7}$$
$$\text{resp. } x_{2i+1} y_{2j+2} - x_{2i+2} y_{2j+1} + x_{2j+1} y_{2j+2} - x_{2j+2} y_{2i+1}. \tag{2.8}$$

□

The full inhomogeneous propagator, which in physics notation would be written $(p_1, \ldots, p_n) \mapsto (\sum a_i p_i + s)^2$ with the p_i and s 4-vectors, becomes in the twistor setup

$$\begin{aligned}(2.9)\quad &\langle (e_1+\sum_{i\geq 3}x_ie_i)\wedge(e_2+\sum_{i\geq 3}y_ie_i),\\ &(c_1e_1^*+c_2e_2^*+\sum_{i\geq 1}a_ie_{2i+1}^*)\wedge(d_1e_1^*+d_2e_2^*+\sum_{i\geq 1}a_ie_{2i+2}^*)\rangle =\\ &\det\begin{pmatrix}c_1 & d_1\\ c_2 & d_2\end{pmatrix}+c_1\sum a_iy_{2i+2}-c_2\sum a_ix_{2i+2}-d_1\sum a_iy_{2i+1}+\\ &d_2\sum a_ix_{2i+1}+\det\begin{pmatrix}\sum a_ix_{2i+1} & \sum a_ix_{2i+2}\\ \sum a_iy_{2i+1} & \sum a_iy_{2i+2}\end{pmatrix}=\\ &\det\begin{pmatrix}\sum a_ix_{2i+1}+c_1 & \sum a_ix_{2i+2}+d_1\\ \sum a_iy_{2i+1}+c_2 & \sum a_iy_{2i+2}+d_2\end{pmatrix}.\end{aligned}$$

Remark 2.3. *In* (2.9)*, our* $\alpha\in\bigwedge^2 V^*$ *is of rank* 2*, i.e. it is decomposible as a tensor and corresponds to an element in* $G(2,V)\subset\mathbb{P}(\bigwedge^2 V^*)$*,* (2.1)*. If we want to add mass to our propagator, we simply replace* α *by* $\alpha+m^2e_1^*\wedge e_2^*$*, yielding* $(\sum a_ip_i+s)^2+m^2$*. The massive* α *represents a point in* $\mathbb{P}(\bigwedge^2 V^*)$ *but not necessarily in* $G(2,V^*)$*.*

3. The Twistor Integral

In this section we take $k=\mathbb{C}$. Consider the maps

$$(3.1)\qquad V\times V-S\xrightarrow{\rho}G(2,V)\xrightarrow{j}\mathbb{P}(\bigwedge^2 V).$$

Here $S=\{(v,w)\ |v\wedge w=0\}$ and $\rho(v,w)=$ 2-plane spanned by v,w.

Lemma 3.1. $V\times V-S/G(2,V)$ *is the principal* $GL_2(\mathbb{C})$*-bundle (frame bundle) associated to the rank* 2 *vector bundle* $\mathcal{W}$ *on* $G(2,V)$ *which associates to* $g\in G(2,V)$ *the corresponding rank* 2 *subspace of* V*.*

Proof. With notation as in (2.1), let $U=\mathrm{Hom}_{\mathbb{C}}(O,I)\subset G(2,V)$. We have

$$(3.2)\qquad \rho^{-1}(U)=\{(z_1,\ldots,z_{2n+2},v_1,\ldots,v_{2n+2})\mid\det\begin{pmatrix}z_1 & z_2\\ v_1 & v_2\end{pmatrix}\neq 0\}.$$

We can define a section $s_U:U\to\rho^{-1}(U)$ by associating to $a:O\to I$ its graph

$$(3.3)\qquad s_U(a):=(1,0,a_1^1,\ldots,a_{2n}^1;0,1,a_1^2,\ldots,a_{2n}^2).$$

Using this section and the evident action of $GL_2(\mathbb{C})$ on the fibres of ρ, we can identify $\rho^{-1}(U)=GL_2(\mathbb{C})\times U$. The fibre $\rho^{-1}(u)$ for $w\in U$ is precisely the set of framings $w=\mathbb{C}z\oplus\mathbb{C}v$ as claimed. ∎

Lemma 3.2. *The canonical bundle* $\omega_{G(2,V)} = \mathcal{O}(-2n-2)$ *where* $\mathcal{O}(-1)$ *is the pullback* $j^*\mathcal{O}_{\mathbb{P}(\bigwedge^2 V)}(-1)$*.*

Proof. The tautological sequence on $G(2,V)$ reads

$$0 \to \mathcal{W} \to V_{G(2,V)} \to V_{G(2,V)}/\mathcal{W} \to 0. \tag{3.4}$$

Here $\mathcal{W}$ is the rank 2 sheaf with fibre over a point of $G(2,V)$ being the corresponding 2-plane in V. One has

$$\Omega^1_{G(2,V)} = \underline{Hom}(V_{G(2,V)}/\mathcal{W}, \mathcal{W}) = (V_{G(2,V)}/\mathcal{W})^\vee \otimes \mathcal{W}. \tag{3.5}$$

By definition of the Plucker embedding j above we have $\mathcal{O}_G(-1) = \bigwedge^2 \mathcal{W}$. The formula for calculating chern classes of a tensor product yields

$$c_1(\Omega^1_G) = c_1((V_{G(2,V)}/\mathcal{W})^\vee)^{\otimes 2} \otimes c_1(\mathcal{W})^{\otimes 2n} = \mathcal{O}_G(-2n-2). \tag{3.6}$$

□

We now fix a point $a \in \mathbb{P}(\bigwedge^2 V^*)$. Upto scale, a determines a non-zero alternating bilinear form on V which we denote by $Q : (x,y) \mapsto \sum_{\nu,\mu} x_\nu Q^{\nu\mu} y_\mu$. By restriction we may view $Q \in \Gamma(G(2,V), \mathcal{O}(1))$. By the lemma $\omega_G \otimes \mathcal{O}(2n+2) \cong \mathcal{O}_G$, so upto scale there is a canonical meromorphic form ξ on $G(2,V)$ of top degree $4n$ with exactly a pole of order $2n+2$ along $Q=0$. We write

$$\xi = \frac{\Xi}{Q^{2n+2}}; \quad 0 \neq \Xi \in \Gamma(G, \omega_G(2n+2)) = \mathbb{C}. \tag{3.7}$$

Lemma 3.3. *We have*

$$H^i(V \times V - S, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 4n+1, 4n+3, 8n+4 \\ (0) & \textit{else} \end{cases}. \tag{3.8}$$

Proof. We compute the dual groups $H^*_c(V \times V - S, \mathbb{Q})$. Note a complex vector space has compactly supported cohomology only in degree twice the dimension. Also, $H^1_c(V - \{0\}) \cong H^0_c(\{0\}) = \mathbb{Q}$. Let $p : S \to V$ be projection onto the first factor. The fibre $p^{-1}(v) \cong \mathbb{C}$ for $v \neq 0$ and $p^{-1}(0) = V$. It follows that

$$H^i_c(S - \{0\} \times V) \cong H^{i-2}_c(V - \{0\}) = (0);\ i \neq 3, 4n+6. \tag{3.9}$$

Now the exact sequence

$$H^i_c(S - \{0\} \times V) \to H^i_c(S, \mathbb{Q}) \to H^i_c(V, \mathbb{Q}) \tag{3.10}$$

yields $H^i_c(S) = \mathbb{Q},\ i = 3, 4n+4, 4n+6$ and vanishes otherwise. Thus, $H^j_c(V \times V - S) = \mathbb{Q};\ j = 4, 4n+5, 4n+7, 8n+8$ and vanishes otherwise. Dualizing, we get the lemma. □

Let $R \subset V \times V$ be the zero locus of the alternating form Q on V defined above. Clearly $S \subset R$.

Lemma 3.4. *Assume the alternating form Q is non-degenerate. Then we have*

$$H^i(V \times V - R, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1, 4n+3, 4n+4 \\ (0) & \textit{else.} \end{cases} \tag{3.11}$$

Proof. Again let $p : R \to V$ be projection onto the first factor. We have $p^{-1}(0) = V$ and $p^{-1}(v) \cong \mathbb{C}^{2n+1}$ for $v \neq 0$. It follows that $H^i_c(R - \{0\} \times V) = (0), i \neq 4n+3, 8n+6$. As before, this yields $H^i_c(R) = \mathbb{Q},\ i = 4n+3, 4n+4, 8n+6$ and zero else. Hence $H^j_c(V \times V - R) = \mathbb{Q},\ j = 4n+4, 4n+5, 8n+7, 8n+8$ and the lemma follows by duality. □

Note that in the case $n = 0,\ \dim V = 2$ we have $S = R$ and the two lemmas give the same information, which also describes the cohomology of the fibres of the map ρ. Namely, $H^i(\rho^{-1}(pt)) = \mathbb{Q},\ i = 0, 1, 3, 4$ and $H^i = (0)$ otherwise.

The form Q induces a quadratic map on $V \times V$ given by $(v, v') \mapsto vQv'$.

Lemma 3.5. *Choose a basis for V and write dv for the evident holomorphic form of degree $4n+4$ on $V \times V$. Then $\mu := dv/Q^{2n+2}$ is homogeneous of degree 0 and represents a non-trivial class in $H^{4n+4}_{DR}(V \times V - R)$.*

Proof. $V \times V - R$ is affine, so we can calculate de Rham cohomology using algebraic forms. There is an evident $\mathbb{G}_m$-action which is trivial on cohomology. Writing a form ν as a sum of eigenforms for this action, we can assume the $\mathbb{G}_m$-action is trivial on ν, which therefore is written $\nu = F dv/Q^{2n+2+N}$ for some $N \geq 0$ and $\deg F = 2N$. Since Q is non-degenerate, we can write $F = \sum_i F_i \partial Q/\partial v_i$. Let $(dv)_i$ be the form obtained by contracting dv against $\partial/\partial v_i$. Then

$$\nu + d\Big(\frac{1}{2n+1+N} \sum F_i (dv)_i / Q^{2n+1+N}\Big) = G dv/Q^{2n+1+N}. \tag{3.12}$$

where G is homogeneous of degree $2(N-1)$. Continuing in this way, we conclude that ν is cohomologous to a constant times dv/Q^{2n+2}. Since by the lemma $H^{4n+4}(V \times V - R) = \mathbb{Q}$, we conclude that $\mu := dv/Q^{2n+2}$ is not exact. □

If one keeps track of the Hodge structure, lemma 3.4 can be made more precise. One gets e.g. $H^{4n+4}(V \times V - R, \mathbb{Q}) \cong \mathbb{Q}(-2n-3)$. For a suitable choice of coordinatizations for the two copies of V and a suitable rational scaling for the chain σ representing a class in $H_{4n+4}(V \times V - R, \mathbb{Q})$ we can write the corresponding period as

$$\int_\sigma d^{2n+2}z \wedge d^{2n+2}v/(\sum z_\mu v_\mu)^{2n+2} = (2\pi i)^{2n+3}. \tag{3.13}$$

Now we make the change of coordinates $v_\mu = \sum_p Q^p_\mu w_p$ and deduce

$$\int_\sigma d^{2n+2}z \wedge d^{2n+2}w/(\sum z_\mu Q^{\mu p} w_p)^{2n+2} = \frac{(2\pi i)^{2n+3}}{\det Q}. \tag{3.14}$$

Here Q is alternating in our case, so $\det Q = \text{Pfaffian}(Q)^2$.

The "Feynman trick" in this context is the integral identity

$$\frac{1}{\prod_{i=1}^{2n+2} A_i} = (2n+1)! \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2}\delta(1-\sum a_i)}{(\sum a_i A_i)^{2n+2}}. \tag{3.15}$$

We apply the Feynman trick with $A_i = \sum_{\mu,p} z_\mu Q_i^{\mu p} w_p$ and integrate over σ

$$\int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}w}{\prod_{i=1}^{2n+2}(\sum_{\mu,p} z_\mu Q_i^{\mu p} w_p)} = \tag{3.16}$$

$$(2n+1)! \int_\sigma d^{2n+2}z \wedge d^{2n+2}w \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2}\delta(1-\sum a_i)}{(\sum a_i(\sum_{\mu,p} z_\mu Q_i^{\mu p} w_p))^{2n+2}} \stackrel{?}{=}$$

$$(2n+1)! \int_{0^{2n+2}}^{\infty^{2n+2}} da_1 \cdots da_{2n+2}\delta(1-\sum a_i) \int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}w}{(\sum_{\mu,p} z_\mu(\sum a_i Q_i^{\mu p})w_p)^{2n+2}} =$$

$$(2n+1)!(2\pi i)^{2n+3} \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2}\delta(1-\sum a_i)}{\text{Pfaffian}(\sum a_i Q_i)^2}.$$

The integral on the right in (3.16) can be rewritten as a projective integral as on the right in (1.4):

$$\int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2}\delta(1-\sum a_i)}{\text{Pfaffian}(\sum a_i Q_i)^2} = \int_\delta \frac{\Omega_{2n+1}}{\text{Pfaffian}(\sum_{i=1}^{2n+2} a_i Q_i)^2}. \tag{3.17}$$

4. Proof of theorem 1.1

To finish the proof of theorem 1.1, we need to understand the chain of integration σ in (3.16). We also need to choose the alternating forms Q_i on the left side of (3.16) so the resulting integral coincides upto a constant with the Feynman integral in the statement of the theorem (1.3).

Put an hermitian metric $||\cdot||$ on V. The induced metric on the bundle of 2-planes defines a submanifold $M \subset V \times V - S$ where M

is the set of pairs $(z, v) \in V \times V - S$ such that $||z|| = ||v|| = 1$ and $\langle z, v\rangle = 0$. M is a $\mathbb{U}_2$-bundle which is a reduction of structure of the $GL_2(\mathbb{C})$ bundle $V \times V - S$. The inclusion $M \subset V \times V - S$ is a homotopy equivalence. In particular, the fibre

$$(R^4\rho_*\mathbb{Z})_w \cong H^4(M_w) = H^4(\mathbb{U}_2) = \mathbb{Z} \cdot [\mathbb{U}_2]. \tag{4.1}$$

($\mathbb{U}_2$ is a compact orientable 4-manifold, so this follows by Poincaré duality.)

For the base, write $G^0 := G(2, V) - \{Q = 0\}$ where $Q \in \bigwedge^2 V^\vee$ is of rank $2n + 2$. G^0 is affine (and hence Stein) of dimension $4n$, so $H^i(G^0, \mathbb{Z}) = (0)$ for $i > 4n$. Let $\rho^0 : V \times V - R \to G^0$ be the GL_2 principal bundle obtained by restriction from ρ. We are interested in the class in $H^{4n+4}(V \times V - R, \mathbb{Q})$ (cf. lemma 3.4) dual to σ. The grassmann is simply connected, so by (4.1), necessarily $R^4\rho_*\mathbb{Z} \cong \mathbb{Z}_G$. Since the fibres of ρ have cohomological dimension 4, we have also

$$\mathbb{Q} = H^{4n+4}(V \times V - R, \mathbb{Q}) \cong H^{4n}(G^0, R^4\rho^0_*\mathbb{Q}) \cong H^{4n}(G^0, \mathbb{Q}). \tag{4.2}$$

It is not hard to show in fact that $H^{4n}(G^0, \mathbb{Q}) = \mathbb{Q} \cdot c_2(\mathcal{W})^n$ where $\mathcal{W}$ is the tautological rank 2 bundle on $G(2, V)$ as in (3.4). The interesting question is what if anything this class has to do with the topological closure of real Minkowski space in $G(2, V)$ which is classically the chain of integration for the Feynman integral.

Recall we have Γ a graph with no self-loops and no multiple edges. External edges will play no role in our discussion, so assume Γ has none. The chain of integration for the Feynman integral is $\mathbb{R}^{4n}$ where n is the loop number of Γ. This vector space is canonically identified with $H := H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4$. In particular, an edge $e \in \mathrm{Edge}(\Gamma)$ yields a functional $e^\vee : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$ associating to a loop ℓ the coefficient of e in ℓ.

To avoid divergences, the theorem is formulated for euclidean propagators. Let $q : \mathbb{R}^4 \to \mathbb{R}$ be $q(x_1, \ldots, x_4) = x_1^2 + \cdots + x_4^2$. The propagators which appear in the denominator of the integral have the form

$$H = H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4 \xrightarrow{e^\vee \otimes id_{\mathbb{R}^4}} \mathbb{R}^4 \xrightarrow{q} \mathbb{R}. \tag{4.3}$$

We take complex coordinates in $\mathbb{C}^4 = \mathbb{R}^4 \otimes \mathbb{C}$ of the form

$$z_1 = x_1 + ix_2,\ z_2 = ix_3 + x_4,\ w_1 = ix_3 - x_4,\ w_2 = x_1 - ix_2; \tag{4.4}$$

$$x_1 = \frac{z_1 + w_2}{2},\ x_2 = \frac{z_1 - w_2}{2i},\ x_3 = \frac{z_2 + w_1}{2i},\ x_4 = \frac{z_2 - w_1}{2}. \tag{4.5}$$

In these coordinates $q = z_1w_2 - z_2w_1$ and the real structure is $\mathbb{R}^4 = \{(z_1, z_2, -\overline{z}_2, \overline{z}_1) \mid z_j \in \mathbb{C}\}$.

Now take real coordinates for $H_1(\Gamma,\mathbb{R})$ and let $(z_1^k, z_2^k, w_1^k, w_2^k),\ k \geq 1$ be the resulting coordinates on $H_{\mathbb{C}}$. It is then the case that for each edge e there are real constants $\alpha_k = \alpha_k(e) \in \mathbb{R}$ not all zero, and the propagator for e is

$$(4.6)\quad \det\begin{pmatrix} \sum_{k\geq 1}\alpha_k z_1^k & \sum_{k\geq 1}\alpha_k z_2^k \\ -\sum_{k\geq 1}\alpha_k \overline{z}_2^k & \sum_{k\geq 1}\alpha_k \overline{z}_1^k \end{pmatrix} = \big|\sum_k \alpha_k z_1^k\big|^2 + \big|\sum_k \alpha_k z_2^k\big|^2.$$

Since the linear functionals associated to the various edges e span the dual space to $H_1(\Gamma,\mathbb{R})$, we see that a positive linear combination of the propagators is necessarily positive definite on $H_{\mathbb{R}}$ (i.e. > 0 except at 0.) Using the cordinates z_i^k, w_i^k we can identify $H_{\mathbb{C}}$ with an open set in $G = G(2, 2n+2)$; namely the point with coordinates z, w is identified with the 2-plane of row vectors

$$(4.7)\qquad \begin{pmatrix} 1 & 0 & z_1^1 & z_2^1 & z_1^2 & z_2^2 & \dots \\ 0 & 1 & w_1^1 & w_2^1 & w_1^2 & w_2^2 & \dots \end{pmatrix}.$$

We throw in two more coordinates z_1^0, z_2^0 (resp. w_1^0, w_2^0) and view the z_j^k (resp. w_j^k) as coordinates of points in $V_{\mathbb{C}} = \mathbb{C}^{2n+2}$. The fact that the set of non-zero matrices of the form $\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$ is a group under multiplication means that the set of non-zero $2 \times (2n+2)$-matrices

$$(4.8)\qquad \begin{pmatrix} z_1^0 & z_2^0 & z_1^1 & z_2^1 & \dots & z_1^n & z_2^n \\ -\overline{z}_2^0 & \overline{z}_1^0 & -\overline{z}_2^1 & \overline{z}_1^1 & \dots & -\overline{z}_2^n & \overline{z}_1^n \end{pmatrix}$$

is closed in G. It is clearly the closure in G of the real Minkowski space whose complex points are given in (4.7). It will be convenient to scale the rows by a positive real scalar and assume $\sum_{j,k} |z_j^k|^2 = 1$, so the resulting locus is compact in $V \times V - R$. We also scale the bottom row by a constant $e^{i\theta}$ of norm 1. The resulting locus

$$(4.9)\quad \sigma := \Big\{\begin{pmatrix} z_1^0 & z_2^0 & z_1^1 & z_2^1 & \dots & z_1^n & z_2^n \\ -e^{i\theta}\overline{z}_2^0 & e^{i\theta}\overline{z}_1^0 & -e^{i\theta}\overline{z}_2^1 & e^{i\theta}\overline{z}_1^1 & \dots & -e^{i\theta}\overline{z}_2^n & e^{i\theta}\overline{z}_1^n \end{pmatrix} \Big| \sum_{j,k} |z_j^k|^2 = 1\Big\} \subset V \times V - R$$

is compact and depends on $4n+4$ real parameters.

Let $Q_e \in \bigwedge^2 V^\vee$ be the form which associates to (4.7) the determinant

$$\det\begin{pmatrix} \sum_{k\geq 1}\alpha_k(e) z_1^k & \sum_{k\geq 1}\alpha_k(e) z_2^k \\ \sum_{k\geq 1}\alpha_k(e) w_1^k & \sum_{k\geq 1}\alpha_k(e) w_2^k \end{pmatrix}.$$

Let $a_e > 0$ be constants, and let $\widetilde{Q} = \sum_e a_e Q_e \in \bigwedge^2 V^\vee$. Finally, let $Q_0 \in \bigwedge^2 V^\vee$ associate to the matrix (4.8) the minor $z_1^0\overline{z}_1^0 + z_2^0\overline{z}_2^0$. It is clear that $Q := Q_0 + \widetilde{Q}$ doesn't vanish on any non-zero matrix of the form (4.8). We conclude:

Proposition 4.1. *Let $G(\mathbb{R}) \subset G$ be the set of points* (4.8)*. Then with Q as above, we have $G(\mathbb{R}) \subset G^0 = G - \{Q = 0\}$.*

The locus σ, (4.9), projects down to $G(\mathbb{R})$ with fibre the group $\mathbb{U}_2$.

Proposition 4.2. *With this choice of σ we have*

$$\int_\sigma \frac{d^{2n+2}z \wedge d^{2n+2}w}{Q^{2n+2}} \neq 0. \tag{4.10}$$

Proof. Let $v_j^{k,\vee}$ be the basis of $V^\vee$ which is dual to the coordinate system z_j^k introduced above. Then one checks that Q as described above is associated to an element

$$Q = \sum_{k=0}^{n} b_k v_1^{k,\vee} \wedge v_2^{k,\vee} \in \bigwedge^2 V^\vee; \quad b_k > 0. \tag{4.11}$$

Applied to the matrix on the right in (4.9),

$$Q(\cdots) = e^{i\theta} \sum_{k=0}^{n} b_k(|z_1^k|^2 + |z_2^k|^2) \tag{4.12}$$

Computing $d^{2n+2}z \wedge d^{2n+2}w$ on the right hand side of (4.9) yields

$$\begin{aligned} &ie^{(2n+2)i\theta} d\theta \wedge \\ &\quad \wedge dz_1^0 \wedge \cdots \wedge dz_2^n \wedge \sum_k \Big((\overline{z}_2^k d\overline{z}_1^k - \overline{z}_1^k d\overline{z}_2^k) \wedge \bigwedge_{j \neq k} (d\overline{z}_1^j \wedge d\overline{z}_2^j) \Big). \end{aligned} \tag{4.13}$$

The crucial point is that the $e^{i\theta}$ factor in the integrand (4.10) cancels. Rescaling we can reduce to the case where all the $b_k = 1$. Integrating over σ yields a $2\pi i$ from the $id\theta$ and then an integral over the volume form of the $4n+3$ sphere $\sum_{k=0}^{n}(|z_1^k|^2 + |z_2^k|^2) = 1$. This is non-zero. □

The proof of theorem 1.1 is now complete. To summarize, given Γ, one uses the change of coordinates (4.4) in order to rewrite the euclidean propagators P_i as determinants of alternating matrices Q_i. One uses the discussion in section 2, particularly formula (2.9) and remark 2.3, to interpret these propagators with external momenta and masses as elements in $\bigwedge^2 V^\vee$, where $V \cong \mathbb{C}^{\mathrm{Edge}(\Gamma)} \cong \mathbb{C}^{2n+2}$. Using (4.6), one sees that a positive linear combination of the Q_i does not vanish on the locus σ defined in (4.9). This means that the integrand on the right in (3.16) has poles only on the boundary of the chain of integration where some of the $a_i = 0$. The integral on the left, given our definition of σ, is a constant (depending only on n) times the euclidean amplitude integral.

References

[B] Bloch, S., Unpublished manuscript on the motive of null vectors for the graph matrix, http://math.uchicago.edu/˜ bloch/publications.html

[BEK] Bloch, S., Esnault, H., Kreimer, D., On motives associated to graph polynomials, Comm. Math. Phys. 267 (2006), no. 1, 181-225.

[BK] Bloch, S., Kreimer, D., Feynman amplitudes and Landau singularities for 1-loop graphs, Comm. Number Theory and Physics 4 (2010), no. 4, 709-753.

[BrK] Broadhurst, D., Kreimer, D., Knots and Numbers in varphi4 Theory to 7 Loops and Beyond, International Journal of Modern Physics C - IJMPC , vol. 6, pp. 519-524, (1995).

[H] Hodges, A., The box integrals in momentum-twistor geometry, arXiv:1004.3323v1 [hep-th].

[IZ] Itzykson, C., Zuber, J.-B., Quantum Field Theory, Dover Publ. (2005).

[K] Kempf, G., On the geometry of a theorem of Riemann, Ann. Math. v. 98, no. 1 (1973), pp. 178-185.

[P] Patterson, E., On the singular structure of graph hypersurfaces, Comm. Number Theory and Physics 4 (2010), no. 4, 659-708.

[T] Todorov, I., Analytic properties of Feynman diagrams in quantum field theory, monographs in natural philosophy 38, Pergamon Press (1971).

5765 S. Blackstone Ave., Chicago, IL 60637, USA, E-mail address: spencer_bloch@yahoo.com