Algebra Placement Exam  
Harris School of Public Policy  
August 31, 2007  
Solutions

1. Linear Equations. (30 points, 5 each) Consider the line defined by the equation  

\[ 2x + 5y - 7 = 0. \]

(a) What is the slope of this line?  
We re-write the equation as  

\[ 5y = -2x + 7, \]

which becomes  

\[ y = -\frac{2}{5}x + \frac{7}{5}. \]

This is now in Slope-Intercept Form, and we see that  \( m = -\frac{2}{5}. \)

(b) Identify any \( x \)- and \( y \)-intercepts of this line.  
The \( y \)-intercept occurs when  \( x = 0 \). Plugging this value into Slope-Intercept Form, we get  

\[ y = -\frac{2}{5} \cdot 0 + \frac{7}{5} = \frac{7}{5}, \]

which makes sense because when we put a line in Slope-Intercept Form (\( y = mx + b \)), the \( b \) represents the \( y \)-intercept.

The \( x \)-intercept occurs when  \( y = 0 \). Plugging this into the original equation for the line, we get  

\[ 2x + 5 \cdot 0 - 7 = 0 \text{ or } 2x - 7 = 0, \]

or, finally,  \( x = \frac{7}{2} \).

(c) Write this line in Slope-Intercept Form.  
In fact, we have already done this above:  \( y = -\frac{2}{5}x + \frac{7}{5} \).

(d) Give the equation of a different line parallel to the given line.  
A line parallel to the given line must have the same slope, which in this case is  \( m = -\frac{2}{5} \). For it to be a different line from the given one, we should make sure that it has a different \( y \)-intercept, so for example, we choose \( b = 1 \), which, using Slope-Intercept Form, gives an equation of  \( y = -\frac{2}{5}x + 1 \).

(e) Give the equation of the line perpendicular to the given line which passes through the point \((3, 4)\).  
If  \( m = -\frac{2}{5} \) is the slope of the given line, then the slope of any perpendicular line is  \( m' = -\frac{1}{m} = \frac{5}{2} \). If the particular perpendicular line must pass through the point \((3, 4)\), then we use Point-Slope Form (\( y - y_1 = m(x - x_1) \)) to get the equation:  

\[ y - 4 = \frac{5}{2}(x - 3). \]

(f) Graph the given line on a reasonable set of axes, being sure to label all intercepts.  
See separate page.

2. Quadratic Equations. (20 points, 5 each) Consider the quadratic defined by the equation  

\[ y = \frac{1}{2}x^2 - 5x - 3. \]

(a) Find the \( x \)-intercepts of this parabola.  
The \( x \)-intercepts occur when  \( y = 0 \). Plugging this value into our equation yields  \( 0 = \frac{1}{2}x^2 - 5x - 3 \), which can be solved using the Quadratic Formula with  \( a = \frac{1}{2} \),  \( b = -5 \), and  \( c = -3 \). The Q. F. yields:  

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 + 4(\frac{1}{2})3}}{2(\frac{1}{2})} = 5 \pm \sqrt{31} \]

(b) Find the vertex of this parabola.  
Given a quadratic in the form  \( y = ax^2 + bx + c \), the vertex occurs when  \( x = -\frac{b}{2a} \). In this case, with  \( b = -5 \) and  \( a = \frac{1}{2} \), we get the \( x \)-coordinate of the vertex to be  \( x = 5 \). Plugging this into the original equation, we find a corresponding \( y \)-coordinate of  \( y = -\frac{31}{2} \).
(c) Write the equation for this parabola in Vertex Form.
We use the method of “completing the square” to write:

\[
y = \frac{1}{2}(x^2 - 10x + 25) - 3 - \frac{25}{2}
\]

(d) Graph the parabola on a reasonable set of axes, being sure to label all points of interest.
See separate page.

3. Inequalities. (16 points, 8 each)
Find all solutions to the following inequalities. Express your answers in interval notation.

(a) \(-\frac{4}{3}x - 6 \leq 20\)
We first add 6 to both sides of the inequality to get \(-\frac{4}{3}x \leq 26\). Next, we multiply both sides of the inequality by \(-\frac{3}{4}\), but since we are multiplying by a negative number, we remember to reverse the sense of the inequality, getting \(x \geq 26 \cdot \left(-\frac{3}{4}\right) = -\frac{39}{2}\). Expressed in interval notation, the solution set to this inequality is \([-\frac{39}{2}, +\infty)\).

(b) \(3x^2 - 10x > 9x - 5\)
Since this is a quadratic inequality, we move all terms to the same side, getting \(3x^2 - 19x + 5 > 0\). With help from the Quadratic Formula, we see that this expression would be equal to zero when

\[
x = \frac{19 \pm \sqrt{(19)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3} = \frac{19 \pm \sqrt{301}}{6}.
\]

In other words, the left-hand side of the inequality factors, and we get:

\[
3 \left(x - \frac{19 + \sqrt{301}}{6}\right) \left(x - \frac{19 - \sqrt{301}}{6}\right) > 0
\]

Since the product of two numbers will be positive precisely when either both are positive or both are negative, we get the solution set \(x < \frac{19 - \sqrt{301}}{6}\) or \(x > \frac{19 + \sqrt{301}}{6}\), which in interval notation is the set \((-\infty, \frac{19 - \sqrt{301}}{6}) \cup (\frac{19 + \sqrt{301}}{6}, +\infty)\).

4. Absolute Value. (16 points, 8 each)
Find all solutions to the following equation and inequality.

(a) \(|2x - 4| = 6\)
Since the absolute value is defined by \(|a| = a\) if \(a \geq 0\) and \(|a| = -a\) if \(a < 0\), we see that we have two potential solutions. The first is given by \(2x - 4 = 6\), in which case \(2x = 10\) or \(x = 5\). The second is given by \(2x - 4 = -6\), which yields \(2x = -2\) or \(x = -1\).

(b) \(|3x + 8| < 12\)
The absolute value inequality \(|a| < b\) is equivalent to the double inequality \(-b < a < b\). In the case of the present inequality, this is \(-12 < 3x + 8 < 12\). Subtracting 8 from all three expressions of the inequality, we get \(-20 < 3x < 4\). Dividing all three expressions by the positive number 3, we get the inequality \(-\frac{20}{3} < x < \frac{4}{3}\).

5. Solving Equations. (12 points, 6 each) Consider the polynomial defined by the equation

\[
y = x^4 - 81.
\]
(a) Factor this polynomial as completely as possible.

Thinking of the term $x^2$ as a single idea, we see that we may use the difference of squares formula to factor $a^2 - b^2 = (a - b)(a + b)$, where in this case, we have $y = (x^2)^2 - (9)^2 = (x^2 - 9)(x^2 + 9)$. Going one step further, we see that $x^2 - 9$ also allows the use of the difference of squares to write $x^2 - 9 = (x - 3)(x + 3)$, so that $y = (x - 3)(x + 3)(x^2 + 9)$. The quadratic $x^2 + 9$ does not factor further.

Another way to begin is to notice that $x = 3$ and $x = -3$ must be roots of the original equation because $3^2 = 81$ and $(-3)^4 = 81$. This implies that $(x - 3)$ and $(x + 3)$ must both factor out of our expression for $y$, and after long division of polynomials, we again get $y = (x - 3)(x + 3)(x^2 + 9)$.

(b) Find all real roots of this polynomial.

The root of a polynomial $y = p(x)$ is a value of $x$ such that $p(x) = 0$. Given the factorization above, we see easily that the two roots of the polynomial are $\pm 3$. (The other two roots are the imaginary numbers $3i$ and $-3i$, but these are not real numbers.)

6. Functions and Graphing. (20 points, 10 each)

For each of the following functions, identify the domain, image, intercepts (if any) and asymptotes (if any), and on a reasonable set of axes, sketch the graph of the function:

(a) $f(x) = \frac{1}{2x^2 + 3x - 2}$

The denominator of this rational function factors as $2x^2 + 3x - 2 = (2x - 1)(x + 2)$ so that our function may be re-written as $f(x) = \frac{1}{(2x - 1)(x + 2)}$. The only problem values for $x$ would be ones that make the denominator equal to 0, and these are $x = -2$ and $x = \frac{1}{2}$, so the domain of $f$ is $\text{Dom}(f) = (-\infty, -2) \cup (-2, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$. There are vertical asymptotes for $f$ at $x = -2$ and at $x = \frac{1}{2}$. There is also a horizontal asymptote for $f$ at $y = 0$. The only intercept is the $y$-intercept, which occurs when $x = 0$, which yields $y = -\frac{1}{2}$, for a $y$-intercept of $(0, \frac{1}{2})$. Finally, the range of $f$ is $\text{Im}(f) = (-\infty, -\frac{8}{25}] \cup (0, +\infty)$. For the graph, see separate page.

(b) $g(x) = \sqrt{9x^2 - 16}$

The restriction on the domain of this function is that we cannot take the square root of a negative number. Thus, we must have $9x^2 - 16 \geq 0$, or $x^2 \geq \frac{16}{9}$. In other words, we must have $x \geq \frac{4}{3}$ or $x \leq -\frac{4}{3}$, which in interval notation gives the domain of $g$ as $\text{Dom}(g) = (-\infty, -\frac{4}{3}] \cup [\frac{4}{3}, +\infty)$. This function has no vertical or horizontal asymptotes (though there are two slant asymptotes of $y = 3x$ and $y = -3x$), and the only intercepts are the obvious $x$-intercepts that occur at $x = \pm \frac{4}{3}$, with corresponding points $(-\frac{4}{3}, 0)$ and $(\frac{4}{3}, 0)$. For the graph, see separate page.

7. Simultaneous Equations. (12 points, 6 each) Consider the following pair of simultaneous equations:

\[
\text{Equation 1: } x + y + 3 = 0 \\
\text{Equation 2: } y = 9 - x^2
\]

(a) Find the two solutions to this pair of equations.

The shortest method of solving this pair of equations is to plug the expression for $y$ in Equation 2 into the proper place in Equation 1, which yields $x + (9 - x^2) + 3 = 0$. This equation simplifies to $-x^2 + x + 12 = 0$, or, if you prefer, $x^2 - x - 12 = 0$. We could use the Quadratic Formula here, but this quadratic has an easy factorization, and we get $(x - 4)(x + 3) = 0$, so that $x = -3$ or $x = 4$. Plugging $x = -3$ into either Equation 1 or 2 yields $y = 0$, and plugging $x = 4$ into either original equation yields $y = -7$. Thus, the two points of intersection of these two curves are: $(-3, 0)$ and $(4, -7)$.
(b) Find the distance in the Cartesian plane between your two solutions. The distance in the plane between the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by the Distance Formula, which is:

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$ 

In our case, this gives, 

$$d = \sqrt{(-3 - 4)^2 + (0 - (-7))^2} = \sqrt{7^2 + 7^2} = \sqrt{98} = 7\sqrt{2}.$$ 

8. Exponentials and Logarithms. (20 points, 5 each)

(a) Solve the following equation: $9^{2x+1} = 27^{5-x}$

We first express each side of the equation in terms of the common base 3, getting $(3^2)^{2x+1} = (3^3)^{5-x}$. The third law of exponents allows us to simplify this to $3^{2(2x+1)} = 3^{3(5-x)}$, and one more step yields $3^{4x+2} = 3^{15-3x}$. Equating the exponents (which we may do since the function $y = 3^x$ is one-to-one), gives $4x + 2 = 15 - 3x$, which simplifies to $7x = 13$ or $x = \frac{13}{7}$.

(b) Solve the following equation: $(2 + \log_5 x)(1 - 2x) = 0$

In order for the product of two real numbers to be zero, we must have one or both of the factors equal to zero. In other words, we may solve the two equations $2 + \log_5 x = 0$ and $1 - 2x = 0$ separately. The first of these becomes $\log_5 x = -2$, which by definition of the logarithm means that $x = 5^{-2} = \frac{1}{25}$. The second of these becomes $2x = 1$, which has the single solution $x = 0$.

(c) Evaluate the following expression: $\log_2 \sqrt{\frac{1}{32}}$

Since $\log_a b = x$ means that $a^x = b$, in our case, we must find an $x$ such that $2^x = \sqrt{\frac{1}{32}} = \frac{1}{\sqrt{32}} = 2^{-5/2}$, and hence $x = -5/2$.

(d) Graph the function $y = 1 + 3^x$ on a reasonable set of axes.

See separate page.

9. Axioms for the Real Numbers. (10 points)

Use the axioms for the real numbers to show that if $a$ and $b$ are any two real numbers, then

$$(a + b)(a - b) = a^2 - b^2.$$ 

A proof with line-by-line justification might go something like this:

\[
\begin{align*}
(a + b)(a - b) &= (a + b)(a - (b)) & \text{by definition of subtraction} \\
&= a \cdot a + a \cdot (-b) + b \cdot a + b \cdot (-b) & \text{by FOIL, or the Distributive Law} \\
&= a^2 - (a \cdot b) + (b \cdot a) - b^2 & \text{by the definition of exponentiation} \\
&= a^2 - (a \cdot b) + (a \cdot b) - b^2 & \text{by the Commutative Law for Multiplication} \\
&= a^2 + 0 - b^2 & \text{by the Additive Inverses Law} \\
&= a^2 - b^2 & \text{by the Additive Identity Law}
\end{align*}
\]

It is worth noting that we have omitted references to the Associative Law, and the simplification from $a \cdot (-b)$ to $-(a \cdot b)$ is standard, but should probably be proved separately.

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