

**Algebra Placement Exam**  
**Harris School of Public Policy**  
**September 17, 2007**

**1. Linear Equations.** (25 points, 5 each)

Consider the three points  $P = (3, -2)$ ,  $Q = (-4, 7)$ , and  $R = (-1, -1)$ .

- (a) **Find the equation of the line passing through points  $P$  and  $Q$ .**

The slope of this line is  $m = \frac{7 - (-2)}{(-4) - 3} = -\frac{9}{7}$ . Using Point-Slope Form with the point  $P$ , we get  $y - (-2) = -\frac{9}{7}(x - 3)$ .

- (b) **Find the  $y$ -intercept of the line passing through points  $P$  and  $Q$ .**

Plugging in the value  $x = 0$  to the equation in part (a), we find  $y = \frac{13}{7}$ , which is the  $y$ -intercept. (And hence the Slope-Intercept Form, though not asked for, is  $y = -\frac{9}{7}x + \frac{13}{7}$ .)

- (c) **Find the equation of the vertical line passing through point  $R$ .**

A vertical line has constant  $x$  value and undetermined  $y$ , so in this case, to pass through the point  $R$ , we must have  $x = -1$ .

- (d) **Find the point of intersection of the lines in parts (a) and (c).**

Clearly, the point of intersection must have  $x = -1$ . Plugging this into the first equation, we get  $y = \frac{22}{7}$ , so that the point is  $(-1, \frac{22}{7})$ .

- (e) **Give the equation of the line perpendicular to the line from part (a) which passes through the point  $R$ .**

The line perpendicular to the line in part (a) must have slope  $+\frac{7}{9}$ . For this line to pass through  $R$ , we use Point-Slope Form once again, getting  $y - (-1) = \frac{7}{9}(x - (-1))$ .

**2. Quadratic Equations.** (20 points, 5 each)

Consider the quadratic defined by the equation

$$y = -9 + 5x - x^2.$$

- (a) **Find all intercepts of this parabola.**

The  $y$ -intercept is easily found by plugging in the value  $x = 0$ . Doing so yields  $y = -9$ . The  $x$ -intercepts would be found by setting  $y = 0$ . Applying the Quadratic Formula then yields  $x = \frac{-5 \pm \sqrt{5^2 - 4(-1)(-9)}}{2(-1)} = \frac{-5 \pm \sqrt{-11}}{-2}$ , and hence there are no real roots and no  $x$ -intercepts.

- (b) **Find the vertex of this parabola.**

Using the shortcut that the  $x$ -coordinate of the vertex must be when  $x = -\frac{b}{2a}$ , we get  $x = \frac{5}{2}$ . Plugging this in, we get  $y = -\frac{11}{4}$  for a vertex of  $(\frac{5}{2}, -\frac{11}{4})$ . Sure enough, completing the square in the original formula gives  $y = -9 + \frac{25}{4} - (x - \frac{5}{2})^2$ , which simplifies to  $y + \frac{11}{4} = -(x - \frac{5}{2})^2$ .

- (c) **Does this parabola open upward or downward? Explain.**

This parabola opens downward since the coefficient of the quadratic term is negative when written in the form  $y = ax^2 + bx + c$ .

- (d) **Graph the parabola on a reasonable set of axes, being sure to label all points of interest.**

See attached page.

**3. Inequalities.** (16 points, 8 each)

**Find all solutions to the following inequalities. Express your answers in interval notation.**

(a)  $4 - x \leq x + 6$

Re-arranging by adding  $x$  to both sides gives  $4 \leq 2x + 6$ . Subtracting 6 from both sides yields  $-2 \leq 2x$ , and dividing both sides by the positive number 2 gives  $-1 \leq x$ . Written in interval notation, we get a solution set of  $[-1, +\infty)$ .

(b)  $7x + 2 < 3x^2$

Moving all terms to the right, we have the inequality  $0 < 3x^2 - 7x - 2$ . Using the Quadratic Formula on the right-hand side, we find roots of  $x = \frac{7 \pm \sqrt{73}}{6}$ , so that we get a factorization on the right, which gives the inequality

$$0 < 3 \left( x - \frac{7 - \sqrt{73}}{6} \right) \left( x - \frac{7 + \sqrt{73}}{6} \right).$$

The expression on the right will be positive when both terms are positive or both are negative (the 3 is always positive). Hence we get a solution set of

$$\left( -\infty, \frac{7 - \sqrt{73}}{6} \right) \cup \left( \frac{7 + \sqrt{73}}{6}, +\infty \right).$$

**4. Absolute Value.** (16 points, 8 each)

(a) **Solve the inequality  $|6 - 2x| > 5$ , and express your answer in interval notation.**

An inequality of the form  $|a| > b$  has solutions  $a > b$  and  $a < -b$ . In this case, the former yields  $6 - 2x > 5$  or  $x < \frac{1}{2}$ , while the latter yields  $6 - 2x < -5$  or  $x > \frac{11}{2}$ . Thus, the full solution set is  $(-\infty, \frac{1}{2}) \cup (\frac{11}{2}, +\infty)$ .

(b) **Graph the equation  $y = |6 - 2x|$ , making sure to indicate intercepts and other points of interest.**

See separate page.

**5. Solving Equations.** (12 points, 6 each) **Consider the polynomial defined by the equation**

$$p(x) = x^3 + 8x^2 + 4x - 3.$$

- (a) **Use the fact that  $p(-1) = 0$  to factor this polynomial as completely as possible.**

Since  $p(-1) = 0$ , we know that  $x + 1$  is a factor of  $p(x)$ . Performing the factorization (using long division of polynomials or otherwise), we get  $p(x) = (x + 1)(x^2 + 7x - 3)$ . Using the Quadratic Formula to factor the last piece, we get:

$$p(x) = (x + 1) \left( x - \frac{-7 - \sqrt{61}}{2} \right) \left( x - \frac{-7 + \sqrt{61}}{2} \right).$$

- (b) **Find all real roots of this polynomial.**

The roots are the values of  $x$  for which  $p(x) = 0$ . Given the factored form of  $p(x)$  in part (a), it is clear that the roots of  $p(x)$  are  $x = -1$ ,  $x = \frac{-7 - \sqrt{61}}{2}$ , and  $x = \frac{-7 + \sqrt{61}}{2}$ .

**6. Functions and Graphing.** (20 points, 10/5/5)

- (a) **Find the domain, image, intercepts, and asymptotes of the function  $f(x) = \frac{1}{\sqrt{x-4}}$ , and graph  $f$  on a reasonable set of axes.**

The domain consists of the values of  $x$  that may be substituted into the expression for  $f(x)$ . In this case, we may not take the square root of a negative number, so we must have  $x \geq 4$ , but we also may not divide by 0, so we cannot have  $x = 4$ . In other words, the domain is  $(4, +\infty)$ .

The image consists of all values that  $f(x)$  may take. In this case, the denominator may be made as large or as small as possible, but it is always positive, so in fact, the image is  $(0, +\infty)$ .

To find the  $y$ -intercept, we would evaluate  $f(0)$ , but 0 is not in the domain of  $f$ , so there is no  $y$ -intercept. To find the  $x$ -intercept, we would set  $f(x) = 0$ , but 0 is not in the image, so there is also no  $x$ -intercept.

There is a vertical asymptote at  $x = 4$ , and there is a horizontal asymptote at  $y = 0$ . For the graph, please see separate sheet.

- (b) **If  $g(x) = x^2 + 2x + 1$ , evaluate the expression  $f(g(3))$ .**

From the expression above,  $g(3) = 3^2 + 2 \cdot 3 + 1 = 16$ , and then  $f(g(3)) = f(16) = \frac{1}{\sqrt{16-4}} = \frac{1}{\sqrt{12}}$ .

- (c) **What is the domain of the function  $f \circ g$ ?**

The function  $f \circ g$  is given by:

$$f \circ g(x) = f(g(x)) = \frac{1}{\sqrt{g(x)-4}} = \frac{1}{\sqrt{x^2+2x-3}} = \frac{1}{\sqrt{(x+3)(x-1)}}.$$

in order for this expression to make sense, we must have  $(x+3)(x-1) > 0$ , and for this to be the case, we must have both factors being positive or both being negative. The set of  $x$  values for which this is the case is  $(-\infty, -3) \cup (1, +\infty)$ .

**7. Simultaneous Equations.** (16 points, 8 each)

**Consider the following pair of simultaneous equations, where  $k$  is a constant:**

$$\begin{array}{ll} \text{Equation 1:} & kx + 3y = 5 \\ \text{Equation 2:} & 4x - 2y = 1 \end{array}$$

- (a) **Determine the value of  $k$  so that this pair of simultaneous equations has no common solution.**

The two lines will not intersect (and hence the two equations will have no common solution) when they are parallel. or in other words, when they have the same slope. The slope of the line in Equation 2 is  $\frac{1}{2}$ . For Equation 1 to have a slope of  $\frac{1}{2}$ , we would need to have  $k = -6$ .

(b) For the value  $k = 1$ , find the unique solution to these two simultaneous equations.

If  $k = 1$ , then Equation 1 may be solved for  $x$  to yield  $x = 5 - 3y$ . Plugging this into Equation 2, we get  $4(5 - 3y) - 2y = 1$ , which simplifies to  $20 - 14y = 1$ , which has solution  $y = \frac{19}{14}$ . Plugging this back into our expression for  $x$ , we get  $x = 5 - 3(\frac{19}{14}) = \frac{13}{14}$ . The unique solution is  $(x, y) = (\frac{13}{14}, \frac{19}{14})$ .

8. Exponents and Logarithms. (20 points, 5 each)

(a) Use the rules of exponents to simplify the following expression:

$$\sqrt[3]{x} \cdot \frac{(x^{-3})^2}{x^{1/2} \cdot x^5}$$

(Express your answer as a single power of  $x$ .)

Recalling that  $\sqrt[3]{x} = x^{1/3}$ , and using the laws of exponents, this simplifies to  $x^{-67/6}$  or  $x^{-11\frac{1}{6}}$ .

(b) Solve the following equation:  $2^{x+2} \cdot 4^{3-x} = 8^x$ .

Re-writing all terms as powers of 2, we get the equation  $2^{x+2} \cdot (2^2)^{3-x} = (2^3)^x$ , which becomes  $2^{x+2} \cdot 2^{6-2x} = 2^{3x}$ . One more step turns this into  $2^{8-x} = 2^{3x}$ , and equating exponents gives the equation  $8 - x = 3x$ , which has solution  $x = 2$ .

(c) Solve the following equation:  $\log_{10} \sqrt{x} = -3$

The definition of the logarithm implies that  $\sqrt{x} = 10^{-3}$ , and hence  $x = 10^{-6} = 0.000001$ .

(d) Graph the equation  $y = \log_2(x - 1)$  on a reasonable set of axes.

See attached page.

9. Axioms for the Real Numbers. (10 points)

Use the axioms for the real numbers to give a formal step-by-step proof that the equation  $-3x + 5 = -7$  has for its only solution the number  $x = 4$ .

$-3x + 5$	$=$	$-7$	given
$(-3x + 5) + (-5)$	$=$	$-7 + (-5)$	by E2 and A4
$-3x + (5 + (-5))$	$=$	$-12$	by A1 and arithmetic
$-3x + 0$	$=$	$-12$	by A4
$-3x$	$=$	$-12$	by A3
$(-\frac{1}{3})(-3x)$	$=$	$(-\frac{1}{3})(-12)$	by E3 and M4
$((-\frac{1}{3})(-3))x$	$=$	$4$	by M1 and arithmetic
$1 \cdot x$	$=$	$4$	by M4
$x$	$=$	$4$	by M3