1. Lines. (20 points, 5 each)

(a) Find the equation of the line $\ell_1$ with slope $-\frac{2}{3}$ that passes through the point $(-7, 4)$. We use Point-Slope Form to find the equation $y - 4 = -\frac{2}{3}(x - (-7))$.

(b) Determine the $y$-intercept of the line $\ell_1$. We rearrange the equation of $\ell_1$ to be in Slope-Intercept Form, we get $y = -\frac{2}{3}x - \frac{2}{3}$, and thus the $y$-intercept is $-\frac{2}{3}$.

(c) Find the equation of the line $\ell_2$ that is perpendicular to $\ell_1$ and that passes through the point $(-7, 4)$. If $\ell_2$ is perpendicular to $\ell_1$, then its slope will be $\frac{3}{2}$. Using Point-Slope Form for $\ell_2$, we find $y - 4 = \frac{3}{2}(x - (-7))$.

(d) Find the equation of the vertical line $\ell_3$ that passes through the point $(-7, 4)$. A vertical line has an equation of the form $x = a$ for some constant $a$, and since the $x$-coordinate of the given point is $-7$, we must have $x = -7$ as the equation for $\ell_3$.

2. Parabolas. (20 points, 5 each) Consider the parabola defined by the equation $y = -\frac{1}{8}x^2 + 3x + 6$.

(a) Find the $x$- and $y$-intercepts of this parabola. The $y$-intercept occurs when $x = 0$. Plugging the value $x = 0$ into the equation yields $y = 6$. The $x$-intercepts occur when $y = 0$. To solve the equation $0 = -\frac{1}{8}x^2 + 3x + 6$, we use the Quadratic Formula, which yields

$$x = \frac{-3 \pm \sqrt{3^2 - 4(-\frac{1}{8})(6)}}{2(-\frac{1}{8})} = 12 \pm 8\sqrt{3}.$$ 

(b) Find the vertex of this parabola. We use the technique of “completing the square” to put this equation into Vertex Form. The final computation yields $y = -\frac{1}{8}(x - 12)^2 + 24$, and thus the vertex of the parabola is the point $(12, 24)$.

(c) Write the equation for this parabola in Factored Form, that is, in the form $y = a(x - x_1)(x - x_2)$. In a sense, we have already done this when we found the $x$-intercepts using the Quadratic Formula in part (a). The final form is $y = -\frac{1}{8}(x - (12 + 8\sqrt{3}))(x - (12 - 8\sqrt{3}))$.

(d) Graph the parabola on a reasonable set of axes, being sure to label all points of interest. See graphs on attached sheet.
3. Inequalities. (20 points, 8/8/4)

Consider the inequality \(12 - x \leq x^2 + 7x - 8\).

(a) Find the solution set for the given inequality.

Algebraic re-arrangement gives the inequality \(0 \leq x^2 + 8x - 20\). The expression on the right-hand side can be factored to yield the inequality \(0 \leq (x - 2)(x + 10)\). In order for the product of two numbers to be positive, we must have both of them positive, which gives \(x \geq 2\), or both of them negative, which gives \(x \leq -10\). Thus, the solution set in interval notation is \((-\infty, -10]\cup[2, +\infty)\).

(b) Graph the two equations \(y = 12 - x\) and \(y = x^2 + 7x - 8\) on the same set of axes.

See attached sheet.

(c) Explain the relationship between your answers to parts (a) and (b).

We observe that the line \(y = 12 - x\) intersects the parabola \(y = x^2 + 7x - 8\) at the points \((-10, 22)\) and \((2, 10)\). When \(x \geq 2\) and when \(x \leq -10\), the graph of the parabola is above that of the line.

4. Absolute Value. (16 points, 8 each)

(a) Find all values of \(x\) that satisfy the equation \(|\frac{1}{2}x + 2| = x + 3\).

If \(\frac{1}{2}x + 2 \geq 0\), which occurs when \(x \geq -4\), the equation becomes \(\frac{1}{2}x + 2 = x + 3\), which simplifies to \(x = -2\). Since this solution occurs in the correct interval (namely, \(-2 \geq -4\)), we accept this as one solution to the equation.

If, on the other hand, \(\frac{1}{2}x + 2 \leq 0\), which occurs when \(x \leq -4\), then the equation becomes \(-\frac{1}{2}x - 2 = x + 3\), which simplifies to \(x = -\frac{10}{3}\). However, this “solution” does not lie in the correct interval since it is not true that \(-\frac{10}{3} \leq -4\).

Thus, the only solution to this equation is \(x = -2\).

(b) Find all values of \(x\) that satisfy the inequality \(|\frac{1}{2}x + 2| < x + 3\).

Similar reasoning to that above shows that if \(x \geq -4\), then the only solutions are when \(x > -2\), all of which fall within the correct interval. And if \(x \leq -4\), then the solutions would be \(x > -\frac{10}{3}\), but none of these fall within the correct interval. Thus, the complete solution set is \(x > -2\), which in interval notation is \((-2, +\infty)\).

5. Polynomials. (16 points, 6/4/4) Consider the polynomial defined by \(p(x) = (x - 2)^5 + 28\).

(a) Write this polynomial in standard form (by multiplying it out or otherwise).

Using the Binomial Theorem, we get

\[ p(x) = x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 + 28. \]

Arithmetic turns this into \(p(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 4\).

(b) Compute \(p(0)\) and \(p(1)\).

Plugging 0 and 1 into either version of the equation for \(p\) yields \(p(0) = -4\) and \(p(1) = 27\).

(c) What can you conclude about the roots of \(p(x)\) based on your answer to (b)?

Since all polynomials are continuous, and since \(p(0) < 0\) and \(p(1) > 0\), we can conclude that \(p(x)\) must have a root somewhere between 0 and 1. (In fact, it turns out that this is the only root, but the there is no immediate way to observe this.)
6. Functions. (16 points, 8 each)

(a) Find the intercepts and asymptotes of the following function: \( f(x) = \frac{x + 1}{(x + 2)(x - 7)(2x + 3)} \)

The y-intercept occurs when \( x = 0 \), which yields \( y = \frac{1}{(2)(-7)(3)} = -\frac{1}{42} \).

The x-intercept occurs when \( y = 0 \), which can only happen when the numerator is 0, namely, when \( x = -1 \).

There is a horizontal asymptote at \( y = 0 \).

There are three vertical asymptotes at \( x = -2 \), \( x = 7 \), and \( x = -\frac{3}{2} \).

(b) Find the domain and range/image of the following function: \( g(x) = \sqrt{\frac{4}{9}x^2 - 16} \)

In order for the definition of \( g(x) \) to make sense, we must have \( \frac{4}{9}x^2 - 16 \geq 0 \). This happens when \( x^2 \geq 36 \), or in other words, when \( |x| \geq 6 \). In interval notation, we can write this domain as \( (-\infty, -6] \cup [6, +\infty) \).

The range of \( g(x) \) is \( [0, +\infty) \) because any non-negative y-value may be obtained.

7. Simultaneous Equations. (16 points, 8 each) Consider the following pair of simultaneous equations:

\begin{align*}
\text{Equation 1:} & \quad 2x + y = -7 \\
\text{Equation 2:} & \quad 3x - y = 22
\end{align*}

(a) Find the solution to this pair of equations.

There are many ways to solve a pair of simultaneous equations. We could solve for one variable and plug it into the other equation, for example. But in this case, the easiest way is to add the two equations to get \( 5x = 15 \), which gives \( x = 3 \). Plugging \( x = 3 \) into either original equation gives \( y = -13 \).

(b) Graph these two lines and illustrate their point of intersection.

See attached sheet.

8. Exponentials and Logarithms. (20 points, 5 each)

(a) Solve the following equation: \( 32 = 4^{-2x+1} \)

Writing everything in base 2, we have \( 2^5 = 2^{-4x+2} \). Taking the log base 2 of both sides or by matching exponents, this gives \( -4x + 2 = 5 \), whose solution is \( x = -\frac{3}{4} \).

(b) Solve the following equation: \( \log_3 (x^3 - 1) = -2 \)

Writing this as an exponential equation yields \( x^3 - 1 = 3^{-2} \) or \( x^3 - 1 = \frac{1}{9} \). Thus, the final answer is \( x = \sqrt[3]{\frac{10}{9}} \).

(c) Simplify the following expression by writing it in the form \( a^k \) for some real number \( k \):

\[ \frac{a^{1/2} \cdot (a^2)^3}{a^{-5} \cdot \sqrt[3]{a^4}} \]

Using the Laws of Exponents, we get

\[ \frac{a^{1/2} \cdot (a^2)^3}{a^{-5} \cdot a^{4/3}} = \frac{a^{1/2} \cdot a^6}{a^{-5} \cdot a^{4/3}} = \frac{a^{13/2}}{a^{-11/3}} = a^{41/2} = a^{61/6} \]

(d) Graph the function \( y = -1 + 21^{-x^2} \) on a reasonable set of axes.

See attached sheet.

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