Algebra Placement Exam Solutions
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1. **Linear Equations I.** (20 points, 5 each)

   Consider the points $A = (-6, 4)$, $B = (3, 1)$ and $C = (7, 13)$ in the Cartesian plane.

   (a) **Find the equation of the line ℓ_1 that passes through A and B.**
   The slope is $m_1 = \frac{1-4}{3-(-6)} = -\frac{1}{3}$, and the Point-Slope Form using the point A yields $y - 4 = -\frac{1}{3}(x - (-6))$, which becomes $y = -\frac{1}{3}x + 2$ in Slope-Intercept Form.

   (b) **Find the equation of the line ℓ_2 that passes through B and C.**
   The slope is $m_2 = \frac{13-1}{7-3} = 3$, and the Point-Slope Form using the point A yields $y - 13 = 3(x - 7)$, which becomes $y = 3x - 8$ in Slope-Intercept Form.

   (c) **Are the lines ℓ_1 and ℓ_2 perpendicular? Explain.**
   Yes, because the slopes are negative reciprocals of each other. That is, $m_1 m_2 = -1$.

   (d) **Find the area of the triangle $\triangle ABC$.**
   Since the line segments AB and BC are in fact perpendicular, we may consider them as the legs of the right triangle $\triangle ABC$. The length of AB is $d(A, B) = \sqrt{(1-4)^2 + (3-(-6))^2} = \sqrt{90}$, and the length of BC is $d(A, B) = \sqrt{(13-1)^2 + (7-3)^2} = \sqrt{160}$. Then the area of the triangle is one half of the product of these lengths, or $\frac{1}{2}\sqrt{90}\sqrt{160} = 60$.
   The other way to do this is to circumscribe the rectangle with vertices $C = (7, 13)$ as well as $D = (-6, 13)$, $E = (-6, 1)$, and $F = (7, 1)$ around the triangle $\triangle ABC$. The area of $\triangle ABC$ will be the area of the circumscribed rectangle (156) minus the sum of the areas of the three triangles $\triangle CDA$ ($58\frac{1}{2}$), $\triangle AEB$ ($13\frac{1}{2}$), and $\triangle BFC$ (24), which gives 60.

2. **Linear Equations II.** (20 points, 5/10/5)

   Consider the following two lines:

Line 1:	$2x + 3y = 4$
Line 2:	$x + 4y = 21$

   (a) **Find the vertical distance between these two lines along the line $x = -4$.**
   The point on ℓ_1 with $x = -4$ is $(-4, 4)$, and the point on ℓ_2 with $x = -4$ is $(-4, \frac{25}{4})$. The vertical distance between these points is $\frac{25}{4} - 4 = \frac{9}{4}$.

   (b) **Find the point of intersection P of these two lines.**
   We solve this system of two equations in two unknowns by multiplying the equation for Line 2 by the constant 2 on both sides to get $2x + 8y = 42$, and then we subtract the equation for Line 1 from this to get $5y = 38$. Thus, $y = \frac{38}{5}$, and substituting this back into the equation for Line 2 yields $x + 4 \cdot \frac{38}{5} = 21$, which gives $x = -\frac{47}{5}$.

   (c) **Find the equation of the line through the point P with slope $m = -\frac{1}{2}$.**
   This is a straightforward exercise in using Point-Slope Form to get $y - \frac{38}{5} = -\frac{1}{2}(x - (-\frac{47}{5}))$, which simplifies to $y = -\frac{1}{2}x + \frac{29}{10}$.

3. Quadratics. (20 points, 10 each) **Consider the parabola defined by the equation**

$$y = \frac{1}{2}x^2 - x - 4.$$

(a) Solve the inequality $\frac{1}{2}x^2 - x - 4 \leq 8$.
We re-arrange the inequality as $\frac{1}{2}x^2 - x - 12 \leq 0$, or, multiplying both sides by 2, as $x^2 - 2x - 24 \leq 0$. The left-hand side factors, so we get $(x-6)(x+4) \leq 0$. The only way for a product to be zero is if one of the factors is zero, so we see that $x = 6$ and $x = -4$ are solutions. The only way for a product of two real numbers to be negative is if one of them is negative and the other is positive. Since $x + 4 > x - 6$ for any x, the term $x + 4$ must be the positive one in this case, and so we need $x + 4 > 0$ and $x - 6 < 0$. This happens exactly when $-4 < x < 6$. Thus, the complete solution set is $-4 \leq x \leq 6$, which can be expressed as the closed interval $[-4, 6]$.

(b) Graph the parabola $y = \frac{1}{2}x^2 - x - 4$, being sure to indicate the vertex and any intercepts.
The y-intercept of this parabola is $y = -4$. To find the x-intercepts, we set $y = 0$ and (by the Quadratic Formula or otherwise) that $x = -2$ or $x = 4$. Completing the square on the original expression for y gives the Vertex Form of the parabola as $y = \frac{1}{2}(x-1)^2 - \frac{9}{2}$, and thus the vertex is $(h, k) = (1, -\frac{9}{2})$.
For the graph, please see attached sheet.

4. Absolute Values and Inequalities. (24 points, 8 each)
Find the solution sets for each of the following equation and inequalities:

(a) $|2x + 4| = 6 + |x|$
Case 1: When $x < -2$, the both expressions in the absolute values are negative, and so we should solve the equation $-(2x + 4) = 6 + (-x)$, which simplifies to $x = -10$. Since this value of x falls within the boundaries of Case 1, we have one solution.

Case 2: When $-2 < x < 0$, the left-hand term in absolute values is positive, which the right-hand term is negative. Thus, we should solve the equation $2x + 4 = 6 + (-x)$, which yields $x = \frac{2}{3}$. However, this value of x falls outside the boundaries of Case 2, so this is not a solution.

Case 3: When $x > 0$, both expressions in absolute values are positive, and so we should solve the equation $2x + 4 = 6 + x$, which yields $x = 2$. Since this value of x falls within the boundaries of Case 3, we have our second solution.

Finally, the values $x = 0$ and $x = -2$ are clearly seen not to be solutions by plugging them into the given equation. Thus, the final set of solutions is $x = -10$ and $x = 2$.

(b) $\dfrac{25}{5 - x} \leq 10$
Case 1: If $x < 5$, then the denominator of the left-hand expression is positive, and so we may multiply both sides of the inequality by $5 - x$ while leaving the direction of the inequality. We get $25 \leq 10(5 - x)$ or $25 \leq 50 - 10x$, which becomes $10x \leq 25$, so that $x \leq \frac{5}{2}$. All of these solutions fall within the range of Case 1, and so this part of the solution set if $(-\infty, \frac{5}{2}]$.

Case 2: If $x > 5$, then the denominator of the left-hand side is negative, so multiplying both sides by $5 - x$ reverses the inequality, and we get $25 \geq 10(5 - x)$ or $25 \geq 50 - 10x$. This becomes $10x \geq 25$ which yields $x \geq \frac{5}{2}$. However, to be in Case 2, we may accept only the solutions that are greater than 5. Thus, this part of the solution set is $(5, +\infty)$.

Finally, $x = 5$ clearly makes the left-hand side undefined, and so the complete solution set is $(-\infty, \frac{5}{2}] \cup (5, +\infty)$.

(c) $|3x - 8| > \dfrac{1}{10}$
This inequality splits into two cases.
Case 1: We solve $3x - 8 > \frac{1}{10}$ by first adding 8 to both sides to get $3x > \frac{81}{10}$ and then dividing both sides by the positive number 3 to get $x > \frac{27}{10}$. This part of the solution set is $(\frac{27}{10}, +\infty)$.

Case 2: We solve $3x - 8 < -\frac{1}{10}$ by first adding 8 to both sides to get $3x < \frac{79}{10}$ and then dividing both sides by the positive number 3 to get $x < \frac{79}{30}$. This part of the solution set is $(-\infty, \frac{79}{30})$. The complete solution set is $(-\infty, \frac{79}{30}) \cup (\frac{27}{10}, +\infty)$.

5. Polynomials. (16 points, 4 each)

Consider the polynomial defined by $p(x) = (x+2)^4 - (x-2)^4$.

(a) Write this polynomial in standard form.

Using the Binomial Theorem or simply using the Distributive Law repeatedly, we get $p(x) = (x^4 + 6x^3 + 24x^2 + 32x + 8) - (x^4 - 6x^3 + 24x^2 - 32x + 8)$. Most of these terms cancel, and we get a simplified form of $p(x) = 16x^3 + 64x$.

(b) What is the degree of this polynomial?

From the standard form above, it is easy to see that the degree of $p(x)$ is 3.

(c) Find the complete factorization of this polynomial (with real coefficients).

Clearly, we have the partial factorization $p(x) = 16x(x^2+4)$. In fact, though, this is the complete factorization since the discriminant of the quadratic term is -12.

(d) Find all real roots of $p(x)$, and explain how you know you have found them all.

The only real root is 0. The roots are the values of x such that $p(x) = 0$, and since we have put $p(x)$ in factored form above, we see that the only way the first factor can be zero is if $x = 0$, and the second factor can never be zero because in fact $x^2 + 3 \geq 3$ for every real value of x.

6. Graphing. (20 points, 15/5)

Consider the following three inequalities:

Inequality 1:	$y \leq 16 - x^2$
Inequality 2:	$y \geq 2x + 1$
Inequality 3:	$\lvert x \rvert > 2$

(a) Graph the set of points in the Cartesian plane that satisfy all three inequalities.

Please see attached graphs.

(b) Find the minimum y-value of any point that satisfies the three inequalities.

Inspecting the graph, the minimum y-value occurs where the parabola intersects the line in the third quadrant. This point is $(-5, -9)$, and thus the answer to the question is -9.

7. Exponentials and Logarithms. (21 points, 7 each)

(a) Put the following four real numbers in increasing order: 120**,** $(\frac{1}{3})^{-5}$**,** $16^{3/2}$**,** $\sqrt[3]{1,000,000}$

First, we compute $(\frac{1}{3})^{-5} = 3^5 = 243$ and $16^{3/2} = 4^3 = 64$ and $\sqrt[3]{1,000,000} = 100$. Thus, the correct order is:

$$16^{3/2}, \quad \sqrt[3]{1,000,000}, \quad 120, \quad (\frac{1}{3})^{-5}$$

(b) Find the value of x that satisfies the following equation:

$$\log_5 (\frac{1}{625}) + \log_3 x = \log_2 32$$

Since $\log_5 (\frac{1}{625}) = -4$ and $\log_2 32 = 5$, we are really solving the equation $\log_3 x = 9$, and the solution is $x = 3^9 = 19,683$.

(c) Find the value(s) of k that satisfy the following equation for all values of $a \neq 0$:

$$\frac{a^{-7/4} \cdot (a^{1/4})^{-6} \cdot a^{k^2}}{a^5 \cdot \sqrt[4]{a^3}} = 1$$

Collecting terms using the rules of exponents gives $a^{k^2-9} = 1$. This means that we should have $k^2 - 9 = 0$, and hence we may have $k = \pm 3$.

8. Algebra. (16 points, 4 each)

Which of the following expressions are algebraically equivalent to $\frac{1}{a+1}$? (For each, answer "yes" or "no.")

(a) $\frac{1}{\sqrt{a^2+1}}$

No! The only value of a for which these two expressions are equal is $a = 0$.

(b) $1 - a + a^2 - a^3 + a^4 - a^5 + \cdots$

No. Although this series does converge to $\frac{1}{a+1}$ when $-1 < a < 1$, the expressions are not equivalent.

(c) $\frac{a-1}{a^2-1}$

No. These expressions are equal for all values of a except $a = 1$.

(d) $(a+1)^{-1}$

Yes. This is the definition of the expression in (d).