1. Linear Equations I. (20 points, 5 each)

Consider the points $A = (-6, 4)$, $B = (3, 1)$ and $C = (7, 13)$ in the Cartesian plane.

(a) Find the equation of the line $\ell_1$ that passes through $A$ and $B$.

The slope is $m_1 = \frac{1-4}{3-(-6)} = -\frac{1}{3}$, and the Point-Slope Form using the point $A$ yields $y - 4 = -\frac{1}{3}(x - (-6))$, which becomes $y = -\frac{1}{3}x + 2$ in Slope-Intercept Form.

(b) Find the equation of the line $\ell_2$ that passes through $B$ and $C$.

The slope is $m_2 = \frac{13-1}{7-3} = 3$, and the Point-Slope Form using the point $A$ yields $y - 13 = 3(x - 7)$, which becomes $y = 3x - 8$ in Slope-Intercept Form.

(c) Are the lines $\ell_1$ and $\ell_2$ perpendicular? Explain.

Yes, because the slopes are negative reciprocals of each other. That is, $m_1m_2 = -1$.

(d) Find the area of the triangle $\triangle ABC$.

Since the line segments $AB$ and $BC$ are in fact perpendicular, we may consider them as the legs of the right triangle $\triangle ABC$. The length of $AB$ is $d(A, B) = \sqrt{(1-4)^2 + (3-(-6))^2} = 9\sqrt{10}$, and the length of $BC$ is $d(A, B) = \sqrt{(13-1)^2 + (7-3)^2} = 160$. Then the area of the triangle is one half of the product of these lengths, or $\frac{1}{2} \cdot 9\sqrt{10} \cdot 160 = 60$.

The other way to do this is to circumscribe the rectangle with vertices $C = (7, 13)$ as well as $D = (-6, 13)$, $E = (-6, 1)$, and $F = (7, 1)$ around the triangle $\triangle ABC$. The area of $\triangle ABC$ will be the area of the circumscribed rectangle (156) minus the sum of the areas of the three triangles $\triangle CDA$ (58), $\triangle AEB$ (13), and $\triangle BFC$ (24), which gives 60.

2. Linear Equations II. (20 points, 5/10/5)

Consider the following two lines:

$$
\text{Line 1:} \quad 2x + 3y = 4 \\
\text{Line 2:} \quad x + 4y = 21
$$

(a) Find the vertical distance between these two lines along the line $x = -4$.

The point on $\ell_1$ with $x = -4$ is $(-4, 4)$, and the point on $\ell_2$ with $x = -4$ is $(-4, \frac{25}{4})$. The vertical distance between these points is $\frac{25}{4} - 4 = \frac{9}{4}$.

(b) Find the point of intersection $P$ of these two lines.

We solve this system of two equations in two unknowns by multiplying the equation for Line 2 by the constant 2 on both sides to get $2x + 8y = 42$, and then we subtract the equation for Line 1 from this to get $5y = 38$. Thus, $y = \frac{38}{5}$, and substituting this back into the equation for Line 2 yields $x + 4 \cdot \frac{38}{5} = 21$, which gives $x = -\frac{47}{5}$.

(c) Find the equation of the line through the point $P$ with slope $m = -\frac{1}{2}$.

This is a straightforward exercise in using Point-Slope Form to get $y - \frac{38}{5} = -\frac{1}{2}(x - (-\frac{47}{5}))$, which simplifies to $y = -\frac{1}{2}x + \frac{29}{10}$.
3. **Quadratics.** (20 points, 10 each) Consider the parabola defined by the equation

\[ y = \frac{1}{2}x^2 - x - 4. \]

(a) **Solve the inequality** \( \frac{1}{2}x^2 - x - 4 \leq 8. \)

We re-arrange the inequality as \( \frac{1}{2}x^2 - x - 12 \leq 0, \) or, multiplying both sides by 2, as \( x^2 - 2x - 24 \leq 0. \) The left-hand side factors, so we get \( (x - 6)(x + 4) \leq 0. \) The only way for a product to be zero is if one of the factors is zero, so we see that \( x = 6 \) and \( x = -4 \) are solutions. The only way for a product of two real numbers to be negative is if one of them is negative and the other is positive. Since \( x + 4 > x - 6 \) for any \( x, \) the term \( x + 4 \) must be the positive one in this case, and so we need \( x + 4 > 0 \) and \( x - 6 < 0. \) This happens exactly when \( -4 < x < 6. \) Thus, the complete solution set is \( -4 \leq x \leq 6, \) which can be expressed as the closed interval \([−4, 6].\)

(b) **Graph the parabola** \( y = \frac{1}{2}x^2-x-4, \) **being sure to indicate the vertex and any intercepts.**

The \( y \)-intercept of this parabola is \( y = -4. \) To find the \( x \)-intercepts, we set \( y = 0 \) and (by the Quadratic Formula or otherwise) that \( x = -2 \) or \( x = 4. \) Completing the square on the original expression for \( y \) gives the Vertex Form of the parabola as \( y = \frac{1}{2}(x - 1)^2 - \frac{9}{2}, \) and thus the vertex is \((h, k) = (1, -\frac{9}{2}).\)

For the graph, please see attached sheet.

4. **Absolute Values and Inequalities.** (24 points, 8 each)

Find the solution sets for each of the following equation and inequalities:

(a) \( |2x + 4| = 6 + |x| \)

Case 1: When \( x < -2, \) the both expressions in the absolute values are negative, and so we should solve the equation \( -(2x + 4) = 6 + (-x), \) which simplifies to \( x = -10. \) Since this value of \( x \) falls within the boundaries of Case 1, we have one solution.

Case 2: When \( -2 < x < 0, \) the left-hand term in absolute values is positive, which the right-hand term is negative. Thus, we should solve the equation \( 2x + 4 = 6 + (-x), \) which yields \( x = \frac{2}{3}. \) However, this value of \( x \) falls outside the boundaries of Case 2, so this is not a solution.

Case 3: When \( x > 0, \) both expressions in absolute values are positive, and so we should solve the equation \( 2x + 4 = 6 + x, \) which yields \( x = 2. \) Since this value of \( x \) falls within the boundaries of Case 3, we have our second solution.

Finally, the values \( x = 0 \) and \( x = -2 \) are clearly seen not to be solutions by plugging them into the given equation. Thus, the final set of solutions is \( x = -10 \) and \( x = 2. \)

(b) \( \frac{25}{5 - x} \leq 10 \)

Case 1: If \( x < 5, \) then the denominator of the left-hand expression is positive, and so we may multiply both sides of the inequality by \( 5 - x \) while leaving the direction of the inequality. We get \( 25 \leq 10(5 - x) \) or \( 5 \leq 50 - 10x, \) which becomes \( 10x \leq 25, \) so that \( x \leq \frac{5}{2}. \) All of these solutions fall within the range of Case 1, and so this part of the solution set if \((−\infty, \frac{5}{2}].\)

Case 2: If \( x > 5, \) then the denominator of the left-hand side is negative, so multiplying both sides by \( 5 - x \) reverses the inequality, and we get \( 25 \geq 10(5 - x) \) or \( 25 \geq 50 - 10x. \) This becomes \( 10x \geq 25 \) which yields \( x \geq \frac{5}{2}. \) However, to be in Case 2, we may accept only the solutions that are greater than \( 5.\) Thus, this part of the solution set is \((5, +\infty).\)

Finally, \( x = 5 \) clearly makes the left-hand side undefined, and so the complete solution set is \((−\infty, \frac{5}{2}] \cup (5, +\infty).\)

(c) \( |3x - 8| > \frac{1}{10} \)

This inequality splits into two cases.

Case 1: We solve \( 3x - 8 > \frac{1}{10} \) by first adding \( 8 \) to both sides to get \( 3x > \frac{81}{10} \) and then dividing both sides by the positive number \( 3 \) to get \( x > \frac{27}{10}. \) This part of the solution set is \((\frac{27}{10}, +\infty).\)
Case 2: We solve \(3x - 8 < -\frac{1}{10}\) by first adding 8 to both sides to get \(3x < \frac{79}{10}\) and then dividing both sides by the positive number 3 to get \(x < \frac{79}{30}\). This part of the solution set is \((-\infty, \frac{79}{30})\).

The complete solution set is \((-\infty, \frac{79}{30}) \cup (\frac{79}{30}, +\infty)\).

5. Polynomials. (16 points, 4 each)

Consider the polynomial defined by \(p(x) = (x + 2)^4 - (x - 2)^4\).

(a) Write this polynomial in standard form.

Using the Binomial Theorem or simply using the Distributive Law repeatedly, we get \(p(x) = (x^4 + 6x^3 + 24x^2 + 32x + 8) - (x^4 - 6x^3 + 24x^2 - 32x + 8)\). Most of these terms cancel, and we get a simplified form of \(p(x) = 16x^3 + 64x\).

(b) What is the degree of this polynomial?

From the standard form above, it is easy to see that the degree of \(p(x)\) is 3.

(c) Find the complete factorization of this polynomial (with real coefficients).

Clearly, we have the partial factorization \(p(x) = 16x(x^2 + 4)\). In fact, though, this is the complete factorization since the discriminant of the quadratic term is \(-12\).

(d) Find all real roots of \(p(x)\), and explain how you know you have found them all.

The only real root is 0. The roots are the values of \(x\) such that \(p(x) = 0\), and since we have put \(p(x)\) in factored form above, we see that the only way the first factor can be zero is if \(x = 0\), and the second factor can never be zero because in fact \(x^2 + 3 \geq 3\) for every real value of \(x\).

6. Graphing. (20 points, 15/5)

Consider the following three inequalities:

**Inequality 1:** \(y \leq 16 - x^2\)

**Inequality 2:** \(y \geq 2x + 1\)

**Inequality 3:** \(|x| > 2\)

(a) Graph the set of points in the Cartesian plane that satisfy all three inequalities.

Please see attached graphs.

(b) Find the minimum \(y\)-value of any point that satisfies the three inequalities.

Inspecting the graph, the minimum \(y\)-value occurs where the parabola intersects the line in the third quadrant. This point is \((-5, -9)\), and thus the answer to the question is \(-9\).

7. Exponentials and Logarithms. (21 points, 7 each)

(a) Put the following four real numbers in increasing order: 120, \((\frac{1}{3})^{-5}\), 16\(^{3/2}\), \(\sqrt[3]{1,000,000}\)

First, we compute \((\frac{1}{3})^{-5} = 3^5 = 243\) and \(16^{3/2} = 4^3 = 64\) and \(\sqrt[3]{1,000,000} = 100\). Thus, the correct order is:

\(16^{3/2}, \sqrt[3]{1,000,000}, 120, (\frac{1}{3})^{-5}\)

(b) Find the value of \(x\) that satisfies the following equation:

\[\log_5 \left(\frac{1}{625}\right) + \log_3 x = \log_2 32\]

Since \(\log_5 \left(\frac{1}{625}\right) = -4\) and \(\log_2 32 = 5\), we are really solving the equation \(\log_3 x = 9\), and the solution is \(x = 3^9 = 19,683\).
(c) Find the value(s) of $k$ that satisfy the following equation for all values of $a \neq 0$:

\[
\frac{a^{-7/4} \cdot (a^{1/4})^{-6} \cdot a^{k^2}}{a^5 \cdot \sqrt[4]{a^3}} = 1
\]

Collecting terms using the rules of exponents gives $a^{k^2-9} = 1$. This means that we should have $k^2 - 9 = 0$, and hence we may have $k = \pm 3$.

8. Algebra. (16 points, 4 each)

Which of the following expressions are algebraically equivalent to $\frac{1}{a + 1}$?

(For each, answer “yes” or “no.”)

(a) $\frac{1}{\sqrt{a^2 + 1}}$

No! The only value of $a$ for which these two expressions are equal is $a = 0$.

(b) $1 - a + a^2 - a^3 + a^4 - a^5 + \ldots$

No. Although this series does converge to $\frac{1}{a+1}$ when $-1 < a < 1$, the expressions are not equivalent.

(c) $\frac{a - 1}{a^2 - 1}$

No. These expressions are equal for all values of $a$ except $a = 1$.

(d) $(a + 1)^{-1}$

Yes. This is the definition of the expression in (d).