Calculus Placement Exam
Harris School of Public Policy
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1. **Limits.** (12 points, 6 each)

   (a) **Evaluate the limit:** $\lim_{x \to -2} (x^2 + 5x - 1)$**.**

   $$\lim_{x \to -2} (x^2 + 5x - 1) = (-2)^2 + 5(-2) - 1 = -7.$$

   (b) **Evaluate the limit:** $\lim_{x \to 4} \frac{x^2 - 7x + 12}{x - 4}$**.**

   $$\lim_{x \to 4} \frac{x^2 - 7x + 12}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x-3)}{x - 4} = \lim_{x \to 4} (x - 3) = 1,$$

   where we are allowed to simplify in the middle step because when evaluating the limit, we do not need to consider $x = 4$.

2. **Continuity.** (12 points, 6 each)

   (a) **Define what it means for a function f to be** *continuous at the point* $x = a$**.**
   The function f is continuous at $x = a$ provided that $\lim_{x \to a} f(x) = f(a)$.

   (b) **Is the function $f(x) = |x - 3|$ continuous at the point $x = 3$? Explain.**
   Yes, this function is continuous at $x = 3$. The left-hand limit is:

   $$\lim_{x \to 3^-} |x - 3| = \lim_{x \to 3^-} -(x - 3) = 0,$$

   and the right-hand limit is:

   $$\lim_{x \to 3^-} |x - 3| = \lim_{x \to 3^+} (x - 3) = 0,$$

   so that the limit of f at $x = 3$ is 0. We note further that $f(3) = 0$, and hence the definition of continuity at a point is satisfied.

3. **Differentiability.** (16 points, 8 each)

   (a) **Define what it means for the function f to be** *differentiable at the point* $x = a$**.**
   The function f is differentiable at $x = a$ provided that the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

   (b) **Use the definition of the derivative to compute $f'(a)$ for the function $f(x) = x^3 - x$.**

   $$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x^3 - x) - (a^3 - a)}{x - a} = \lim_{x \to a} \frac{(x^3 - a^3) - (x - a)}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)(x^2 + xa + a^2 - 1)}{x - a} = \lim_{x \to a} (x^2 + xa + a^2 - 1) = 3a^2 - 1 \end{aligned}$$

4. Derivatives. (24 points, 8 each)

For each of the following, find $\frac{dy}{dx}$:

(a) $y = (x^2 + 3x)^{5/2}$

$$\frac{dy}{dx} = \frac{5}{2}(x^2 + 3x)^{3/2} \cdot (2x + 3)$$

(b) $y = \dfrac{x}{1 + \ln\ x}$

$$\frac{dy}{dx} = \frac{(1 + \ln\ x) \cdot 1 - x \cdot (\frac{1}{x})}{(1 + \ln\ x)^2} = \frac{\ln x}{(1 + \ln x)^2}$$

(c) $y^2 + xy + x^2 = e^x$

Use implicit differentiation as follows: $2y \cdot \dfrac{dy}{dx} + (x \cdot \dfrac{dy}{dx} + y \cdot 1) + 2x = e^x$.

Simplifying yields $\dfrac{dy}{dx} \cdot (x + 2y) = e^x - 2x - y$, which becomes $\dfrac{dy}{dx} = \dfrac{e^x - 2x - y}{x + 2y}$.

5. Tangent Lines. (12 points)

Find the equation of the line tangent to the curve $y = \frac{1}{2}x^3 - \frac{1}{x}$ when $x = 2$.

Differentiation yields $\dfrac{dy}{dx} = \dfrac{3}{2}x^2 + \dfrac{1}{x^2}$. When $x = 2$, we get $y = \frac{7}{2}$ and $\frac{dy}{dx} = \frac{25}{4}$.

Putting this information into Point-Slope Form gives the equation $y - \dfrac{7}{2} = \dfrac{25}{4}(x - 2)$.

In Slope-Intercept Form, this becomes $y = \dfrac{25}{4}x - 9$.

6. Optimization. (12 points)

Suppose the sum of three positive numbers is 24 and that the first number is three times the second one. Find (with explanation) the maximum product of these three numbers.

If x, y, and z are the three numbers, then the conditions of the problem give $x = 3y$ and $x+y+z = 24$. Combining these two equations gives $z = 24 - 4y$, which means that we may write our product as a function of just one variable as follows:

$$f(y) = x \cdot y \cdot z = (3y) \cdot y \cdot (24 - 4y) = 72y^2 - 12y^3$$

The natural domain of this function is clearly $[0, 6]$ since the numbers are required to be positive. (If $y > 6$, then we would have $z < 0$.) We look for other critical points of this function by differentiating and find that $f'(y) = 144y - 36y^2$. This derivative always exists, so there are no singular points, and the derivative is 0 when $36y(4 - y) = 0$, which yields the two stationary points of $y = 0$ and $y = 4$. Since the maximum must occur at one of the critical points, we simply compare the values of f at 0, 4, and 6 and find that $f(0) = 0$, $f(4) = 384$, and $f(6) = 0$. Hence the maximum occurs when $y = 4$, and the three numbers are 12, 4, and 8.

7. Analysis of Functions. (30 points, 5 each)

Consider the function $f(x) = x^2 \cdot e^{-x}$.

- **(a) What is the domain of f?**

  The domain is all real numbers.

- **(b) Identify all critical points of f,**

  There are no endpoints. To find stationary and singular points, we compute

  $$f'(x) = x^2 \cdot e^{-x} \cdot (-1) + e^{-x} \cdot (2x) = e^{-x}(2x - x^2).$$

  This derivative always exists, so there are no singular points. The derivative equals zero when $x = 0$ ad $x = 2$, so those are the stationary points, and that completes the list.

- **(c) Find the intervals on which f is increasing and decreasing.**

  Checking the sign of the terms in the factored form of f' shows that f is decreasing on the interval $(-\infty, 0)$, increasing on the interval $(0, 2)$, and decreasing on the interval $(2, +\infty)$.

- **(d) Determine all local maxima and minima of f.**

  By the First Derivative Test, f has a local minimum at $x = 0$ and a local maximum at $x = 2$.

- **(e) Find the intervals on which f is concave up and concave down.**

  To find potential inflection points and to determine the intervals on which f is concave up or down, we compute the second derivative as follows:

  $$f''(x) = e^{-x} \cdot (2 - 2x) + (2x - x^2) \cdot e^{-x} \cdot (-1) = e^{-x}(x^2 - 4x + 2).$$

  The second derivative equals zero when $x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$. Checking the sign of the term $x^2 - 4x + 2$ (since e^{-x} is always positive), we see that f is concave up on the interval $(-\infty, 2 - \sqrt{2})$, concave down on $(2 - \sqrt{2}, 2 + \sqrt{2})$ an concave up on $(2 + \sqrt{2}, +\infty)$.

- **(f) Determine all inflection points of f.**

  Since the concavity changes at both $x = 2 \pm \sqrt{2}$, these are both inflection points.

8. Asymptotes. (15 points, 5 each)

Consider the function $f(x) = \dfrac{3 - x}{2x + 4}$.

- **(a) Identify any horizontal asymptotes of f.**

  There is a horizontal asymptote at $y = -\frac{1}{2}$.

- **(b) Identify any vertical asymptotes of f.**

  There is a vertical asymptote at $x = -2$.

- **(c) For each vertical asymptote, evaluate the left- and right-hand limits of f at the asymptote.**

  $$\lim_{x \to (-2)^-} \frac{3 - x}{2x + 4} = -\infty$$

  $$\lim_{x \to (-2)^+} \frac{3 - x}{2x + 4} = +\infty$$