Calculus Placement Exam
Harris School of Public Policy
September 21, 2015

You have ninety minutes for this exam. No books, notes, calculators, or other aids are allowed. Please answer in the blue books provided, and please make sure to include your name and UCID number on all work submitted. Finally, there is partial credit to be awarded, so please SHOW YOUR WORK!

1. Limits (15 points, 5 each)
   For each of the following, evaluate the limit, or state that the limit does not exist (no justification required):
   (a) \( \lim_{x \to 2} \frac{x}{\sqrt{x^2 + 5}} \)
   (b) \( \lim_{x \to -4} \frac{x}{\sqrt{x + 4}} \)
   (c) \( \lim_{x \to 1} \frac{1 - x}{1 - x^2} \)

2. Differentiation (21 points, 7 each)
   Differentiate the following functions. You may use any theorems.
   (a) \( f(x) = \frac{x^4 - 1}{2x} \)
   (b) \( g(x) = (2x + 1) \cdot e^{\sqrt{x}} \)
   (c) \( h(x) = \left( \frac{1}{2} x^2 + \ln x \right)^{-1/3} \)

3. Properties of the Derivative (20 points, 10 each)
   Let \( f : [-4, 4] \to \mathbb{R} \) be a continuous function, and suppose \( f'(x) \) exists for all \( x \) in the interval \((-4, 4)\) with the following values:
   \[
   f'(x) = \begin{cases} 
   2, & \text{if } -4 < x < 1 \\
   4 - 2x, & \text{if } 1 \leq x < 4 
   \end{cases}
   \]
   (a) Identify the intervals on which \( f \) is increasing and decreasing.
   (b) Identify (by their \( x \) values) any local maxima and minima of \( f \) on \([-4, 4]\).

4. Optimization I (20 points)
   With justification, find all global and local maxima and minima of the function
   \( f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 7 \)
   on the interval \([-2, 2]\).
5. Implicit Differentiation (16 points, 8 each)
Consider the equation \( xe^y + x^2 e^{-y} = 6 \).

(a) Use implicit differentiation to find \( \frac{dy}{dx} \).

(b) Determine \( \frac{dy}{dx} (2,0) \) and find the equation of the line tangent to the curve at this point.

6. Analysis of Functions I (18 points, 3 each)
Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by the formula:
\[ f(x) = \ln(2x^2 + 1). \]

(a) Compute \( f'(x) \).

(b) Identify all intervals on which \( f \) is increasing and decreasing.

(c) Identify all local maxima and minima of \( f \).

(d) Compute \( f''(x) \).

(e) Identify all intervals on which \( f \) is concave up and concave down.

(f) Identify any inflection points of \( f \).

7. Analysis of Functions II (20 points, 8/6/4)
For constants \( a, b > 0 \), consider the function
\[ f(x) = \frac{ax^2}{x^2} - \frac{b}{x} \]
on the domain \((0, +\infty)\).

(a) Suppose \( f(x) \) has an inflection point at \((1, -8)\). Find the values of \( a \) and \( b \).

(b) Given the values of \( a \) and \( b \) from part (a), identify all local maxima and minima of \( f(x) \).

(c) Sketch a graph of \( y = f(x) \) on an appropriately scaled set of axes.

8. Partial Derivatives and Optimization II (22 points, 6/6/10)
Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by the formula:
\[ f(x, y) = x^2y - 1. \]

(a) If \( z = f(x, y) \), sketch the level curves of \( f \) for the \( z \)-values \(-1, 0, 1 \) and \( 2 \) on the same graph.

(b) Compute the gradient of \( f \), namely \( \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \).

(c) Use Lagrange multipliers to find the maximum value of \( f \) subject to the constraint \( x + y = 1 \).

9. Definite Integrals (12 points)
Evaluate the following definite integral and sketch a graph of what this integral represents:
\[ \int_1^4 (3 + 2\sqrt{x}) \, dx \]

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