Definitions and Theorems 12

Limits at Infinity, Infinite Limits, and Asymptotes

Definition. (Limits at Infinity)

1. $\lim_{x \to +\infty} f(x) = L$ means that:
   given any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that if $x > N$, then $|f(x) - L| < \varepsilon$.
2. $\lim_{x \to -\infty} f(x) = L$ means that:
   given any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that if $x < -N$, then $|f(x) - L| < \varepsilon$.

Definition. (Infinite Limits)

1. $\lim_{x \to a^+} f(x) = +\infty$ means that:
   given any $N \in \mathbb{N}$, there is some $\delta > 0$ such that if $0 < x - a < \delta$, then $f(x) > N$.
2. $\lim_{x \to a^+} f(x) = -\infty$ means that:
   given any $N \in \mathbb{N}$, there is some $\delta > 0$ such that if $0 < x - a < \delta$, then $f(x) < -N$.
3. $\lim_{x \to a^-} f(x) = +\infty$ means that:
   given any $N \in \mathbb{N}$, there is some $\delta > 0$ such that if $0 < a - x < \delta$, then $f(x) > N$.
4. $\lim_{x \to a^-} f(x) = -\infty$ means that:
   given any $N \in \mathbb{N}$, there is some $\delta > 0$ such that if $0 < a - x < \delta$, then $f(x) < -N$.

Theorem.

If $n > 0$, then $\quad \lim_{x \to \pm\infty} \frac{1}{(x-a)^n} = 0, \quad \lim_{x \to a^-} \frac{1}{x-a} = -\infty \quad$ and $\quad \lim_{x \to a^+} \frac{1}{x-a} = +\infty.$

Theorem.

If $a > 0$, then $\lim_{x \to -\infty} a^x = 0$.

Theorem.

If $a > 0$, then $\lim_{x \to 0^+} \log_a x = -\infty$.

Definition. (Asymptotes)

1. The line $y = L$ is a *horizontal asymptote* of the function $f(x)$ provided that:
   $$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$
2. The line $y = mx + b$ is a *slant (or oblique) asymptote* of the function $f(x)$ provided that:
   $$\lim_{x \to +\infty} [f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x \to -\infty} [f(x) - (mx + b)] = 0$$
3. The line $x = a$ is a *vertical asymptote* of the function $f(x)$ provided that:
   $$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$