Definitions and Theorems 3

Exponents

Definition. Let $a > 0$.

- If n a natural number n, we define $a^n = a \cdot a \cdot a \cdots a$ (n copies of a).
- We further define $a^0 = 1$.
- If n is still a natural number, we define $a^{-n} = \frac{1}{a^n}$.
- If q is another natural number, we define $a^{1/q} = \sqrt[q]{a}$.
- If p is an integer and q is a natural number, we define $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.
- Finally, if x is an irrational real number, we define a^x in such a way as to make the function $f(x) = a^x$ continuous.

Theorem (Laws of Exponents). Let $a > 0$. For any real numbers n and m, we have:

1. $a^n \cdot a^m = a^{n+m}$
2. $\dfrac{a^n}{a^m} = a^{n-m}$
3. $(a^n)^m = a^{n \cdot m}$

Logarithms

Definition. Let $a > 0$. If $a^x = b$, then we define $\log_a b = x$.

Theorem (Laws of Logarithms). Let $a > 0$. For any $x, y > 0$, we have:

1. $\log_a (x \cdot y) = \log_a x + \log_a y$
2. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a (x^n) = n \cdot \log_a x$

Relationship between Exponentials and Logarithms

Theorem. Let $a > 0$.

- If $x \in \mathbb{R}$, then $\log_a (a^x) = x$.
- If $x > 0$, then $a^{(\log_a x)} = x$.