Definitions and Theorems 4

Limits

Definition. A function f is said to have **limit** L **at** $x = a$ provided that:
given any $\varepsilon > 0$, there is some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
When the limit exists, we denote this by $\lim_{x \to a} f(x) = L$.

Definition. A function f is said to have **right-hand limit** L **at** $x = a$ provided that:
given any $\varepsilon > 0$, there is some $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.
When the limit exists, we denote this by $\lim_{x \to a^+} f(x) = L$.

Definition. A function f is said to have **left-hand limit** L **at** $x = a$ provided that:
given any $\varepsilon > 0$, there is some $\delta > 0$ such that if $0 < a - x < \delta$, then $|f(x) - L| < \varepsilon$.
When the limit exists, we denote this by $\lim_{x \to a^-} f(x) = L$.

Theorem. If $c \in \mathbb{R}$, then $\lim_{x \to a} c = c$.

Theorem. If $a \in \mathbb{R}$, then $\lim_{x \to a} x = a$.

Theorem. Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then:

(a) $\lim_{x \to a} (f(x) + g(x)) = L + M$.

(b) $\lim_{x \to a} (f(x) - g(x)) = L - M$.

(c) $\lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M$.

(d) $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$, provided that $M \neq 0$.

Continuity

Definition. A function f is said to be **continuous at** $x = a$ provided that: $\lim_{x \to a} f(x) = f(a)$.
(Note that, in particular, this implies that the limit must exist!)

Definition. A function f is said to be **continuous on the interval** $[a, b]$ provided that:

(a) f is continuous at x, for each $x \in (a, b)$.

(b) $\lim_{x \to a^+} f(x) = f(a)$.

(c) $\lim_{x \to b^-} f(x) = f(b)$.