Definitions and Theorems 5

Differentiation

Definition. Let f be a function defined on the interval (a, b), and let $c \in (a, b)$. We say that f is **differentiable at** $x = c$ provided that the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. When this limit exists, we denote it $f'(c)$ and call this number the **derivative of** f **at** $x = c$. We say that f is **differentiable on** (a, b) provided that f is differentiable at each $c \in (a, b)$.

Theorem. If $k \in \mathbb{R}$ and $f(x) = k$, then f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = 0$.

Theorem. If $m, b \in \mathbb{R}$ and $f(x) = mx + b$, then f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = m$.

Theorem. (Power Rule) If $f(x) = x^n$, then f is differentiable for every $x > 0$ and $f'(x) = n \cdot x^{n-1}$.
Examples:

- If $f(x) = x^0 = 1$, then $f'(x) = 0$.
- If $f(x) = x^1 = x$, then $f'(x) = 1$.
- If $f(x) = x^2$, then $f'(x) = 2x$.
- If $f(x) = x^3$, then $f'(x) = 3x^2$.
- If $f(x) = x^{-1} = \frac{1}{x}$, then $f'(x) = (-1)x^{-2} = -\frac{1}{x^2}$.
- If $f(x) = x^{-2} = \frac{1}{x^2}$, then $f'(x) = (-2)x^{-3} = -\frac{2}{x^3}$.
- If $f(x) = x^{1/2} = \sqrt{x}$, then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Theorem. Suppose f and g are both defined on (a, b) and differentiable at $c \in (a, b)$. Then:

(i) (Sum Rule) $f + g$ is differentiable at $x = c$ and $(f + g)'(c) = f'(c) + g'(c)$.
(ii) (Difference Rule) $f - g$ is differentiable at $x = c$ and $(f - g)'(c) = f'(c) - g'(c)$.
(iii) (Constant Multiple Rule) If $k \in \mathbb{R}$, then $k \cdot f$ is differentiable at $x = c$ and $(k \cdot f)'(c) = k \cdot f'(c)$.

Theorem. (Product Rule) Suppose f and g are both defined on (a, b) and differentiable at $c \in (a, b)$. Then $f \cdot g$ is differentiable at $x = c$ and

$$(f \cdot g)'(c) = f(c) \cdot g'(c) + g(c) \cdot f'(c).$$

Theorem. (Quotient Rule) Suppose f and g are both defined on (a, b) and differentiable at $c \in (a, b)$, and suppose that $g(c) \neq 0$. Then $\frac{f}{g}$ is differentiable at $x = c$ and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c) \cdot f'(c) - f(c) \cdot g'(c)}{[g(c)]^2}.$$