Elementary Number Theory
Math 175, Section 30
Autumn Quarter 2008
Written Exercises from Weeks 3–4

Exercise 0.0.1 Find the unique prime factorizations of the following positive integers. Feel free to use technological assistance.

- 1,000,000
- 1,000,001
- 4,999,999
- 123,456,789

Exercise 0.0.2 Use the Euclidean Algorithm to find the greatest common divisors (a, b) for the following a and b:

- $a = 233$ and $b = 377$
- $a = 3,657,329$ and $b = 1,348,867$
- $a = d - 1$ and $b = d + 1$ for some $d \in \mathbb{Z}$.

Exercise 0.0.3 Let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

Define addition and multiplication in $\mathbb{Z}[\sqrt{-5}]$ as follows:

$$(a + b\sqrt{-5}) + (c + d\sqrt{-5}) = (a + c) + (b + d)\sqrt{-5}$$

$$(a + b\sqrt{-5}) \cdot (c + d\sqrt{-5}) = (ac - 5bd) + (ad + bc)\sqrt{-5}$$

Define the norm map $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ as follows:

$$N(a + b\sqrt{-5}) = a^2 + 5b^2$$

We say that an element $z \in \mathbb{Z}[\sqrt{-5}]$ is *prime* if there do not exist $x, y \in \mathbb{Z}[\sqrt{-5}]$, with both $N(x) > 1$ and $N(y) > 1$, such that $z = xy$.

1. Show that $\mathbb{Z}[\sqrt{-5}]$ is a commutative ring with identity (see Algebra script if you need the definition).
2. Show that $N(xy) = N(x)N(y)$ for any $x, y \in \mathbb{Z}[\sqrt{-5}]$.
3. Show that the only elements $x \in \mathbb{Z}[\sqrt{-5}]$ with $N(x) \leq 1$ are $x = -1, 0, 1$.
4. Show that $\mathbb{Z}[\sqrt{-5}]$ does not have unique factorization into primes by showing that $x = 6$ can be factored into primes in two distinct ways.