Autumn Quarter 2006
Math 199, Section 51: Introduction to Analysis and Linear Algebra
John Boller
Homework 6
Due: WEDNESDAY, November 22, 2006
Final Draft

1. Read Chapter 2, "Linear Algebra," especially Sections 2.1–2.3.
2. (*) Prove that every element of a vector space has a unique additive inverse.
3. (*) Prove the Cancellation Law for Vector Spaces: If $c \cdot \mathbf{u} = c \cdot \mathbf{v}$ and $c \neq 0$, then $\mathbf{u} = \mathbf{v}$.
4. Let V be a vector space over F, and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in V$ be a finite set of vectors in V. Show that $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is a subspace of V.
5. Let V be a vector space over F, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Show that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if there is some scalar $c \in F$ such that $\mathbf{v}_1 = c \cdot \mathbf{v}_2$ (or vice versa).
6. Fix $k \in \mathbb{N}$, and consider the real vector space
$$V_k = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \mid a_i \in \mathbb{R}, 0 \leq i \leq k\}.$$
   (a) (*) Show that V_k is a subspace of $\mathbb{R}[x]$.
   (b) Find $dim(V_k)$ by exhibiting a basis and proving that it is so.
7. (*) Find a basis for the vector space $V = \mathbb{R}^4$ that includes the vectors $\mathbf{u} = (1, 2, 3, 4)$ and $\mathbf{v} = (0, -1, -1, -1)$, and prove that your set of vectors is indeed a basis.
8. Let p be a prime number, and let $F = \mathbb{Z}_p$ be the field of congruence classes of integers modulo p (see HW #2.12). Let $V = F^2$ be the vector space of ordered pairs of elements of F.
   (a) How many vectors are there in the vector space V?
   (b) How many 0-dimensional subspaces does V have? Explain.
   (c) How many 2-dimensional subspaces does V have? Explain.
   (d) How many 1-dimensional subspaces does V have? Explain. Write down a complete list in the case $p = 7$.

   This is the interesting part of the question. Note that there is an exact numerical answer that depends on p. Hints: 1. Consider the subspaces $W = \text{span}\{\mathbf{w}\}$ where $\mathbf{w} \in V$ is a single vector. 2. Are any of the subspaces from Hint 1 the same?

   The reason we use a *finite* field for this question is that it allows us to consider the complete list of such subspaces. Note that if we had asked the same question for $V = \mathbb{R}^2$, the number of 1-dimensional subspaces would clearly be infinite.
9. Let $dim(V) = n$, and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in V. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set if and only if $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.